‘A teacher affects eternity; he can never tell where his influence stops.’

– Henry Adams

MATHEMATICS

LESSON PLAN
GRADE 11 TERM 3

A MESSAGE FROM NECT

NATIONAL EDUCATION COLLABORATION TRUST (NECT)

Dear Teachers

This learning programme and training is provided by the National Education Collaboration Trust (NECT) on behalf of the Department of Basic Education (DBE). We hope that this programme provides you with additional skills, methodologies and content knowledge that you can use to teach your learners more effectively.

WHAT IS NECT?

In 2012 our government launched the National Development Plan (NDP) as a way to eliminate poverty and reduce inequality by the year 2030. Improving education is an important goal in the NDP which states that 90% of learners will pass Maths, Science and languages with at least 50% by 2030. This is a very ambitious goal for the DBE to achieve on its own, so the NECT was established in 2015 to assist in improving education.

The NECT has successfully brought together groups of people interested in education so that we can work collaboratively to improve education. These groups include the teacher unions, businesses, religious groups, trusts, foundations and NGOs.

WHAT ARE THE LEARNING PROGRAMMES?

One of the programmes that the NECT implements on behalf of the DBE is the 'District Development Programme'. This programme works directly with district officials, principals, teachers, parents and learners; you are all part of this programme!

The programme began in 2015 with a small group of schools called the Fresh Start Schools (FSS). Curriculum learning programmes were developed for Maths, Science and Language teachers in FSS who received training and support on their implementation. The FSS teachers remain part of the programme, and we encourage them to mentor and share their experience with other teachers.

The FSS helped the DBE trial the NECT learning programmes so that they could be improved and used by many more teachers. NECT has already begun this scale-up process in its Universalisation Programme and in its Provincialisation Programme.

Everyone using the learning programmes comes from one of these groups; but you are now brought together in the spirit of collaboration that defines the manner in which the NECT works. Teachers with more experience using the learning programmes will deepen their knowledge and understanding, while some teachers will be experiencing the learning programmes for the first time.

Let's work together constructively in the spirit of collaboration so that we can help South Africa eliminate poverty and improve education!

www.nect.org.za

CONTENTS

PROGRAMME ORIENTATION

Welcome!

The NECT FET Mathematics Learning Programme is designed to support teachers by providing:

- Lesson Plans
- Trackers
- Resource Packs
- Assessments and Memoranda
- Posters.

This Mathematics Learning Programme takes care of most of the planning required to teach FET Mathematics. However, it is important to remember that although the planning has been done for you, preparation is key to successful teaching. Ensure that you put aside adequate time to properly prepare to teach each topic.

Also remember that the most important part of preparation is ensuring that you develop your own deep conceptual understanding of the topic. Do this by:

- working through the lesson plans for the topic
- watching the recommended video clips at the end of the topic
- completing all the worked examples in the lesson plans
- completing all activities and exercises in the textbook

If, after this, a concept is still not clear to you, read through the section in the textbook or related teacher's guide, or ask a colleague for assistance. You may also wish to search for additional teaching videos and materials online.

Orientate yourself to this Learning Programme by looking at each component, and by taking note of the points that follow.

TERM 3 TEACHING PROGRAMME

1. In line with CAPS, the following teaching programme has been planned for FET Mathematics for Term 3:

Grade 10		Grade 11		Grade 12	
Topic	No. of weeks	Topic	No. of weeks	Topic	No. of weeks
Analytical Geometry	2	Measurement	1	Euclidean Geometry	2
Finance and Growth	2	Euclidean Geometry	3	Statistics	2
Statistics	2	Trigonometry	2	Counting and probability	2
Trigonometry	1.5	Finance, growth and decay	2		
Euclidean Geometry	1	Probability	2		
Measurement	1.5				

2. Term 3 lesson plans and assessments are provided for ten weeks for Grades 10 and 11.
3. Term 3 lesson plans and assessments are provided for six weeks for Grade 12
4. Each week includes 4,5 hours of teaching time, as per CAPS.
5. You may need to adjust the lesson breakdown to fit in with your school's timetable.

LESSON PLAN STRUCTURE

The Lesson Plan for each term is divided into topics. Each topic is presented in exactly the same way:

TOPIC OVERVIEW

1. Each topic begins with a brief **Topic Overview**. The topic overview locates the topic within the term, and gives a clear idea of the time that should be spent on the topic. It also indicates the percentage value of this topic in the final examination, and gives an overview of the important skills and content that will be covered.

2. The **Lesson Breakdown Table** is essentially the teaching plan for the topic. This table lists the title of each lesson in the topic, as well as a suggested time allocation. For example:

	Lesson title	Suggested time (hours)
1.	Revision	1,5
2.	Gradient and average gradient	1

3. The **Sequential Table** shows the prior knowledge required for this topic, the current knowledge and skills to be covered, and how this topic will be built on in future years.
   - Use this table to think about the topic conceptually.
     - Looking back, what conceptual understanding should learners have already mastered?
     - Looking forward, what further conceptual understanding must you develop in learners, in order for them to move on successfully?
   - If learners are not equipped with the knowledge and skills required for you to continue teaching, try to ensure that they have some understanding of the key concepts before moving on.
   - In some topics, a revision lesson has been provided.
4. The **NCS Diagnostic Reports**. This section is potentially very useful. It lists common problems and misconceptions that are evident in learners' NSC examination scripts. This Lesson Plan aims to address these problem areas, but it is also a good idea for you to keep these in mind as you teach a topic.
5. The A**ssessment of the Topic** section outlines the formal assessment requirements as prescribed by CAPS for Term 3.

Grade	Assessment requirements for Term 3 (as prescribed in CAPS)
10	Two tests
11	Two tests
12	One test and one preliminary examination

6. The glossary of **Mathematical Vocabulary** provides an explanation of each word or phrase relevant to the topic. In some cases, an explanatory sketch is also provided. It is a good idea to display these words and their definitions or sketches somewhere in the classroom for the duration of the topic. It is also a good idea to encourage learners to copy down this table in their free time, or alternately, to photocopy the Mathematical Vocabulary for learners at the start of the topic. You should explicitly teach the words and their meanings as and when you encounter these words in the topic.

INDIVIDUAL LESSONS

1.. Following the **Topic Overview**, you will find the **Individual Lessons.** Each lesson is structured in exactly the same way. The routine within the individual lessons helps to improve time on task, and therefore, curriculum coverage.

2. In addition to the lesson title and time allocation, each lesson plan includes the following:

   A. Policy and Outcomes. This provides the CAPS reference, and an overview of the objectives that will be covered in the lesson.

   B. Classroom Management. This provides guidance and support as you plan and prepare for the lesson.

   - Make sure that you are ready to begin your lesson, have all your resources ready (including resources from the Resource Pack), have notes written up on the chalkboard, and are fully prepared to begin.
   - Classroom management also suggests that you plan which textbook activities and exercises will be done at which point in the lesson, and that you work through all exercises prior to the lesson.
   - In some cases, classroom management will also require you to photocopy an item for learners prior to the lesson, or to ensure that you have manipulatives such as boxes and tins available.

   ***The Learner Practice Table*.** This lists the relevant practice exercises that are available in each of the approved textbooks.

   - It is important to note that the textbooks deal with topics in different ways, and therefore provide a range of learner activities and exercises. Because of this, you will need to plan when you will get learners to do the textbook activities and exercises.
   - If you feel that the textbook used by your learners does not provide sufficient practice activities and exercises, you may need to consult other textbooks or references, including online references.
   - The *Siyavula* Open Source Mathematics textbooks are offered to anyone wishing to learn mathematics and can be accessed on the following website: https://www.everythingmaths.co.za/read

C. ***Conceptual Development*:**

This section provides support for the actual teaching stages of the lesson.

Introduction: This gives a brief overview of the lesson and how to approach it. Wherever possible, make links to prior knowledge and to everyday contexts.

Direct Instruction: Direct instruction forms the bulk of the lesson. This section describes the teaching steps that should be followed to ensure that learners develop conceptual understanding. It is important to note the following:

- Grey blocks talk directly to the teacher. These blocks include teaching tips or suggestions.
- Teaching will often be done by working through an example on the chalkboard. These worked examples are always presented in a table. This table may include grey cells that are teaching notes. The teaching notes help the teacher to explain and demonstrate the working process to learners.
- As you work through the direct instruction section, and as you complete worked examples on the chalkboard, ensure that learners copy down:
  - formulae, reference notes or explanations
  - the worked examples, together with the learner's own annotations.
- These notes then become a reference for learners when completing examples on their own, or when preparing for examinations.
- At relevant points during the lesson, ensure that learners do some of the Learner Practice activities as outlined at the beginning of each lesson plan. Also, give learners additional practice exercises as homework. Ensure that learners are fully aware of your expectations in this respect.

D. ***Additional Activities / Reading***. This section provides you with web links related to the topic. Get into the habit of visiting these links as part of your lesson preparation. As a teacher, it is always a good idea to be more informed than your learners. If possible, organise for learners to view video clips that you find particularly useful.

TRACKER

1. A Tracker is provided for each grade for each term. The Trackers relate directly to the Lesson Plans, and are CAPS compliant in terms of content and time.
2. You can use the Tracker to document your progress. This helps you to monitor your pacing and curriculum coverage. If you fall behind, make a plan to catch up.
3. Fill in the Tracker on a daily or weekly basis.

4. At the end of each week, try to reflect on your teaching progress. This can be done with the HoD, with a subject head, with a colleague, or on your own. Make meaningful notes about what went well and what didn't. Use the reflection section to reflect on your teaching, the learners' learning and to note anything you would do differently next time. These notes can become an important part of your preparation in the following year.

RESOURCE PACK, ASSESSMENT AND POSTERS

1. A Resource Pack with printable resources has been provided for each term.
2. These resources are referenced in the lesson plans, in the Classroom Management section.
3. Two posters have been provided as part of the FET Mathematics Learning Programme for Term 3.
4. Ensure that the posters are displayed in the classroom.
5. Try to ensure that the posters are durable and long-lasting by laminating it, or by covering it in contract adhesive.
6. Note that you will only be given these resources once. It is important for you to manage and store these resources properly. You can do this by:
   - Writing your school's name on all resources
   - Sticking resource pages onto cardboard or paper
   - Laminating all resources, or covering them in contact paper
   - Filing the resource papers in plastic sleeves once you have completed a topic.
7. Add other resources to your resource file as you go along.
8. Note that these resources remain the property of the school to which they were issued.

ASSESSMENT AND MEMORANDUM

In the Resource Pack you are provided with assessment exemplars and memoranda as per CAPS requirements for the term. For Term 3, the Resource Pack contains two tests and memoranda for Grade 10, and contains two tests and memoranda for Grade 11. One test, with memorandum, is provided for Grade 12. If your learners write a common test, you could use the test provided for revision.

CONCLUSION

Teacher support and development is a complex process. For successful Mathematics teachers, certain aspects of this Learning Programme may strengthen your teaching approach. For emerging Mathematics teachers, we hope that this Learning Programme offers you meaningful support as you develop improved structures and routines in your classroom, develop deeper conceptual understanding in your learners, and increase curriculum coverage.

Term 3, Topic 1: Measurement

TOPIC OVERVIEW

TOPIC OVERVIEW

A

- This topic is the first of five topics in Term 3.
- This topic runs for 1 weeks (4,5 hours).
- It is presented over one lesson of one hours – the rest of the time will be used to practise questions on the Grade 10 work.
- Euclidean Geometry and Measurement counts 33% of the final Paper 2 examination.
- Although measurement does not form a large part of the final assessment, do not allow learners to deem it unimportant. An understanding of both surface area and volume is essential in our daily lives.

Breakdown of topic into 1 lesson:

	Lesson title	**Suggested time (hours)**
1	Revision of concepts	4,5

B

SEQUENTIAL TABLE

GRADE 10 and Senior phase	GRADE 11	GRADE 12
LOOKING BACK	CURRENT	LOOKING FORWARD
• Revise the volume and surface areas of right-prisms and cylinders. • Study the effect on volume and surface area when multiplying any dimension by a constant factor k. • Calculate the volume and surface areas of spheres, right pyramids and cones	• Revise all Grade 10 work.	Measurement is not a stand-alone topic in Grade 12. The skills acquired up to and including Grade 10 & 11 will be used to solve problems involving volume and surface area of solids. These problems are often linked to Trigonometry.

C

WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are few issues pertaining directly to Measurement as it is incorporated into other areas in Grade 12.

While revising Measurement, give learners the opportunity to use manipulatives, particularly if they are struggling to understand the concepts. The use of manipulatives can assist learners in development of the required concepts and the development of an understanding of the formulae used.

D

ASSESSMENT OF THE TOPIC

- CAPS formal assessment requirements for Term 3:
  - Two tests.
- Two tests, with memoranda, are provided in the Resource pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of diagrams of 3-dimensional objects related to a word problem and learners need to find surface area and/or volume of the solids.
- Monitor each learner’s progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	Explanation
2D	2-dimensional
3D	3-dimensional
polygon	A 2D shape in which all the sides are made up of line segments. A polygon is given a name depending on the number of sides it has. For example: A 5-sided polygon is called a pentagon
solid	An object that occupies space (3-dimensional)
prism	A solid object with two identical ends and flat sides The shape of the ends gives the prism a name, example 'triangular prism' ● The cross section is the same all along its length ● The sides are parallelograms
right prism	A right prism is a geometric solid that has a polygon as its base and vertical sides perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and sides are right angles
face	A flat surface of a 3D object
cube	3D object (solid) with six equal square faces
rectangular prism	3D object (solid) with six rectangular faces
triangular prism	3D object (solid) with two equal triangular faces (one is the base) and three rectangular faces
cylinder	3D object (solid) with two equal circular faces (one is the base) and one rectangle (curved)
sphere	Round solid figure, with every point on its surface equidistant from its centre
cone	3D object (solid) which tapers from a circular base to a point
pyramid	3D object (solid) with a square or triangular base and sloping sides formed by triangles that meet in a point at the top
net	A 2D shape that, when folded, forms a 3D object

surface area	Area taken up by the net of a 3D object. The sum of the area of all the faces
volume	The space taken up by a 3D object. To find volume, the area of the base is multiplied by the perpendicular height. This only works for right prisms
capacity	The amount a 3D shape can hold. It is directly linked to volume
frustum	The portion of a cone or pyramid which remains after its upper part has been cut off by a plane parallel to its base, or which is intercepted between two such planes
hemisphere	Half a sphere

TERM 3, Topic 1, Lesson 1

REVISION OF GRADE 10 WORK

Suggested lesson duration: 4,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	34
Lesson Objectives By the end of the lesson, learners will have revised: • Volume and surface area of 3D objects, including the cube, rectangular prism, triangular prism, cylinder, cone, sphere and pyramid.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. You will need Resources 1 and 2 from the Resource Pack.
4. Write the lesson heading on the board before learners arrive.
5. Write work on the chalkboard before the learners arrive. For this lesson draw the seven 3D objects. Do not label the 3D objects.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

C

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev	205	1 2 Rev	182 183 185	Qu's 1 Qu's	195 197 200	8.1 8.2 8.3 8.4 Rev	260 264 266 269 270	7.1 7.2 7.3 7.4 7.5 7.6	310 314 317 321 325 327

D

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. By Grade 11, learners should have a good understanding of surface area and volume. These concepts have been covered since the Senior Phase.
2. Only a small part of this lesson will be to revise the four basic 3D objects covered since Grade 8 and Grade 9. This will take the form of a short discussion and summary of the formulae. Most of the lesson will be devoted to revising the three new 3D objects introduced in Grade 10. If you feel that your learners need to spend more time working with the basic 3D objects, adjust the lesson accordingly.
3. Plan the time given with care and according to your learners' ability. You may want to do one or two examples with them first, and then give them the opportunity to practice on their own before doing another example. You should use your textbook as a guide. Some textbooks only have one or two exercises whereas others have more. It is always useful for you, as teacher, to have more than one textbook so that you can supplement activities in the learners' textbook with activities from other textbooks where necessary.
4. Whenever learners are working on their own, make an effort to assist them by asking directed questions, rather than just giving them the solution.

TOPIC 1, LESSON 1: REVISION OF GRADE 10 WORK

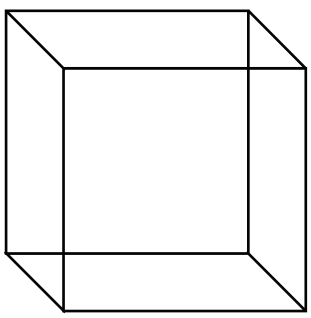
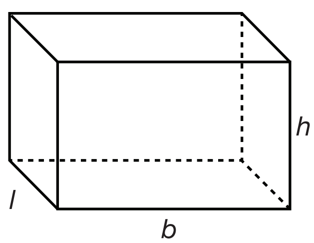
DIRECT INSTRUCTION

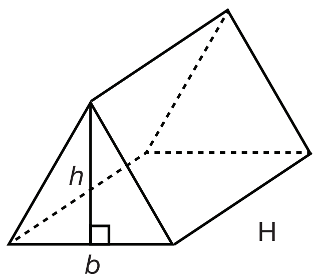
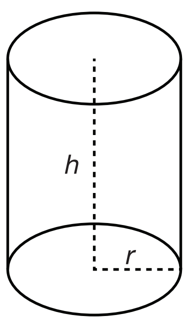
1. Start the lesson by asking learners what the difference is between surface area and volume.
   Surface area: The area taken up by the net of a 3D object. The sum of the area of all the faces.
   Volume: The space taken up by a 3D object.

2. Point to the four 3D objects and ask learners to name them. As they do so, label them on the board.
   (cube, rectangular prism, triangular prism and cylinder).

3. Draw a table like the one below (the sketches are not required as you have them on the board) and complete it with learners. Ask learners for the formulae to populate the table. Remind learners that it is not always a good idea to learn the formulae for surface area as the formulae assume a closed shape. In Grade 11 the shapes asked will almost always be complex shapes (at least 2 shapes combined).

   Remind learners that the volume formula for right prisms and the right cylinder is always 'Area of base multiplied by perpendicular height'.

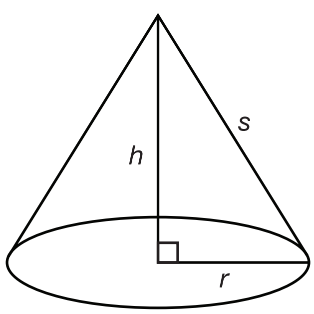
Shape	Surface Area	Volume
Cube 	$6l^2$	$l \times l \times l = l^3$
Rectangular prism 	$2lb + 2bh + 2lh$	$l \times b \times h$

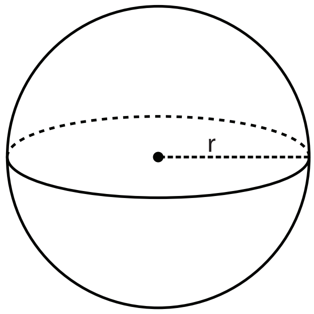
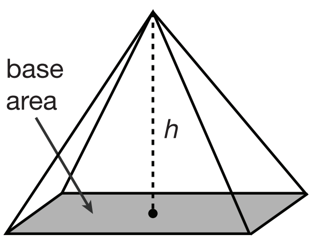
Triangular prism 	Sum of the areas of the 2 triangles and 3 rectangles.	$(\frac{1}{2}b \times h) \times H$ Note: The first 'h' represents the height of the triangle which is required to find the area of the base. The 2nd 'H' represents the height of the prism.
Cylinder 	$2\pi r^2 + 2\pi rh$	$\pi r^2 \times h$

4. Point out the other three sketches on the chalkboard. Ask if anyone can tell you the names of the 3D objects. Fill them in as they are given (cone, sphere, and pyramid).

5. Tell learners that the formulae will always be given when questioned on one of these shapes.

6. Give learners a summary of the formulae. Learners should copy the summary in their exercise books for reference purposes.

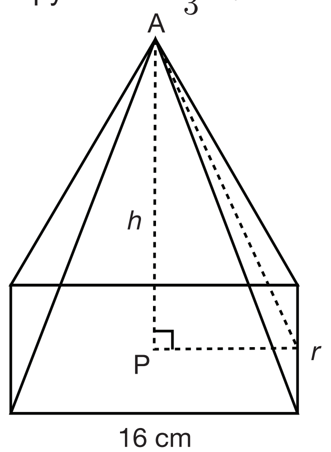
Shape	**Surface Area**	**Volume**
Cone 	$\pi rs + \pi r^2$ (the slant height is sometimes named l)	$\frac{1}{3}\pi r^2 \times h$

Sphere 	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Pyramid 	Sum of the areas of: ● the base and ● the triangles* * the number of triangles depends on the type of base	$\frac{1}{3}$ (area of base) $\times h$ (Remember that the base could be any polygon but a square, rectangle or triangle are generally used).

7. Ask learners if they have any questions before moving on to doing some examples with them that they can expect to get at this level.

8. As you do each of the following examples from past papers, use questioning to get as much input from learners as possible. Ask learners questions and encourage discussion. Stop after each example down in their exercise books.

Note: These questions are provided in the Resource Pack for those who have access to photocopying facilities.

Example 1	Teaching notes:
A pyramid with a square base with a side length of 16cm is sketched below. P lies on the square base directly below A. The volume of the pyramid is 640cm^3. Volume of a pyramid $=\frac{1}{3}Ah$. 	Ask learners: *'What does 'directly below' tell you?'* (it is the perpendicular height)
a) Show that the perpendicular height of the pyramid, AP, is 7,5cm.	Remind learners that if a question's instruction is 'show that' (or prove that) and the answer is given in the question, the learners must be very careful how they answer. The learners may NOT use the answer given in their solution. Tell the learners to rather imagine that the question has asked, 'find the height' and use the fact that the height is given as an opportunity to check their answer. Ask learners: *How will you 'find' the height?* (Volume is given so use the volume formula – which is given – and fill in all known details then solve for height).

b) Hence, determine the total surface area of the pyramid. NSC NOV 2017	Discuss the word 'hence' with learners. Tell them that 'hence' means they should use the information from the previous question to solve this question. Point out that they could do this question using the information given even if they did not manage to get that right. Ask: *How will you find surface area of the pyramid?* (Find the area of the square base and the area of the 4 triangles and add them together). Ask: *How will you find the height of the triangle?* (The height of the triangle is the slant height of the pyramid which can be found using the theorem of Pythagoras).

Solution:

a) Volume $= \frac{1}{3}Ah$

$$640cm^3 = \frac{1}{3}(16cm)(16cm)h$$

$$\frac{640cm^3}{\frac{1}{3}(16cm)(16cm)} = h$$

$$\therefore h = 7{,}5cm$$

b) Slant height:

$$s^2 = 8^2 + (7{,}5)^2$$

$$s^2 = 120{,}25$$

$$\therefore s = 10{,}9658\ldots.$$

Surface area = area of base + 4 triangles

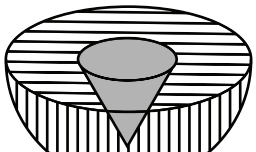
$$= l^2 + 4\left(\frac{1}{2}bh\right)$$

$$= (16)^2 + 2(16)(10{,}9658\ldots)$$

$$= 606{,}91cm^2$$

Example 2

A solid metallic hemisphere has a radius of 3cm. It is made of metal A. To reduce its weight, a conical hole is drilled into the hemisphere (as shown in the diagram) and it is completely filled with a lighter metal B. The conical hole has a radius of 1,5cm and a depth of $\frac{8}{9}$ cm.



Volume of a sphere $= \frac{4}{3}\pi r^3$

Volume of a cone $= \frac{1}{3}\pi r^2 h$

Calculate the ratio of the volume of metal A to the volume of metal B.

EXEMPLAR 2013

Teaching notes:

Ask: *What is a hemisphere?*
(Half a sphere).
Ask: *How will we find the ratio of the metals?*
(Find the volume of the hemisphere and the volume of the cone then subtract the cone from the hemisphere).
Ask: *Is all the information required for the formula available?*
(Yes – the radius of the sphere is given, and the radius and height of the cone is given).
Point out to learners that π is part of both formulae and is therefore part of the answer to both volumes. As we need to find the ratio of the two, it will be easier if we leave the answers in π form as they will simplify in the division process of finding the ratio.

Solution:

Volume of the hemisphere:

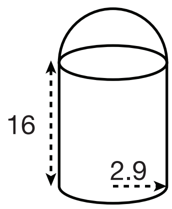
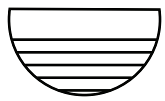
$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$
$$V = \frac{2}{3}\pi\,(3cm)^3$$
$$V = 18\pi cm^3$$

Volume of the cone:

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\,(1{,}5cm)^2\left(\frac{8}{9}\right)$$
$$V = \frac{2}{3}\pi cm^3$$

$$\therefore \frac{V\ of\ metal\ A}{V\ of\ metal\ B} = \frac{18\pi cm^3}{\frac{2}{3}\pi cm^3} = 26$$

$\therefore$ the ratio is 26:1

Example 3	**Teaching notes:**
A cylindrical aerosol can has a lid in the shape of a hemisphere that fits exactly on the top of the can. The height of the can is 16cm and the radius of the base of the can is 2,9cm. Volume of a sphere $= \frac{4}{3}\pi r^3$ Surface area of a sphere $= 4\pi r^2$  Figure 1  Figure 2 a) Calculate the surface area of the can with the lid in place, as shown in FIGURE 1.	Ask: *What should you always be careful of in a surface area question?* (Is the 3D object a closed one or not – there may be a face missing) Ask: *Is there is an issue in this case?* (Yes- the top circle of the cylinder will not be included because it is part of the inside of the can). Ask: *What is the surface area of the can is made up of?* (The base circle, the circular part of the cylinder and the hemisphere).
b) If the lid is 80% filled with a liquid, as shown in FIGURE 2, calculate the volume of the liquid in the lid. NSC NOV 2015	Ask: *What can you tell me about the relationship between capacity and volume?* (Capacity is directly linked to volume). Point out, however, that the question is still about volume, so they will not need to know the conversions from volume to capacity.

Solution:

a) Surface area $= \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2)$

$= \pi(2,9)^2 + 2\pi(2,9)(16) + 2\pi(2,9)^2$

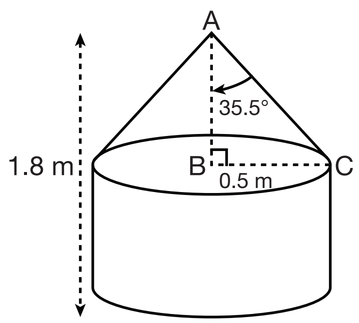
$= 370,8cm^2$

b) Volume of the lid $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{2}{3}\pi(2,9)^3$

$= 57,08cm^3$

80% of $57,08cm^3 = 40,86cm^3$

Example 4	**Teaching notes:**
The diagram below shows a water tank which is made up of a cylinder and a cone having equal radii. The height of the tank is 1,8m and the radius is 0,5m. The angle between the perpendicular height, AB, and the slant height, AC, of the conical section is 35,5°. Volume of a cone $= \frac{1}{3}\pi r^2 h$ Total surface area of cone $= \pi r^2 + \pi rs$ Volume of cylinder $= \pi r^2 h$ Total surface area of cylinder $= 2\pi r^2 + 2\pi rh$ 	
a) Calculate the perpendicular height, AB, of the cone.	Ask: *What other topic of mathematics will need to be used to find the perpendicular height AB*? (Trigonometry – it is an unknown side in a right-angled triangle where a side and an angle are given). Ask: *Which trig ratio will be used?* (According to the angle named, we have the opposite side and the adjacent side is required, therefore, tan will be used).

b) When the tank is full, an electric pump switches on, and pumps the water from the tank into an irrigation system at a rate of $0,52m^3/h$. The pump automatically switches off when the tank is $\frac{1}{4}$ full. Calculate how long, in hours, the pump feeds water into the irrigation system. NSC NOV 2016	Point out that capacity is a key point again (as in the last example). Ask: *What is capacity linked to?* (Volume). The volume of the 3D object needs to be found. Ask: *Do you have all the information for the variables required?* (Yes – we have radius and height of the cylinder – they were given. We also have radius and height of the cone – the height was just found). *Ask: What is the height of the cylinder part?* (1,8m minus the height of the cone which is the answer to a). Ask: *What fraction of the tank is being pumped out ?* $\left(\frac{3}{4}\right)$. We will need to find $\frac{3}{4}$ of the volume. Once this has been found it can be used along with the rate given to find the number of hours taken.

Solution:

a) $\tan 35,5^o = \frac{0,5}{\text{AB}}$

$\text{AB} \tan 35,5^o = 0,5$

$\text{AB} = \frac{0,5}{\tan 35,5^o}$

$\therefore \text{AB} = 0,7m$

b) Volume of tank:

$V = \frac{1}{3}\pi r^2 h + \pi r^2 h$

$V = \frac{1}{3}\pi (0,5)^2 (0,7) + \pi (0,5)^2 (1,1)$

$V = 1,04m^3$

Volume of water being pumped out:

$\frac{3}{4}(1,04m^3) = 0,78m^3$

Time taken by the pump:

$\frac{0,78m^3}{0,52m^3} = 1,5$

The time taken is 1,5 hours

9. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

10. Give learners an exercise to complete with a partner.

11. Walk around the classroom as learners do the exercise. Support learners where necessary.

E ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=V5xp1i3A2w8
(Surface area of a composite shape)

https://www.youtube.com/watch?v=bolTauMy4B4
(Volume of a composite shape)

https://www.youtube.com/watch?v=jklxFO0MFw8
(Volume and surface area of a composite shape)

https://www.youtube.com/watch?v=5tPMvXYPzHc
(Frustum of a cone) Note: A frustum is not mentioned in CAPS but as it is a 3D object derived from 3D objects that are mentioned it could come up in a question. More importantly though is that learners are introduced to a new concept which will enrich their learning.

https://www.youtube.com/watch?v=bN9jHdWJzaI
(Pyramid frustum)

Term 3, Topic 2: Euclidean Geometry

TOPIC OVERVIEW

TOPIC OVERVIEW

A

- This topic is the second of five topics in Term 3.
- This topic runs for three weeks (13,5 hours).
- It is presented over seven lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 13,5 hours). For example, one lesson in this topic could take three school lessons. Plan according to your school's timetable.
- Euclidean Geometry and Measurement counts 33% of the final Paper 2 examination.
- The following link to a 'Guide to Circle Geometry' is worth a read before teaching the section http://learn.mindset.co.za/sites/default/files/resourcelib/emshare-topic-overview-asset/maths-11-3-guide-circle-geometry.pdf
- There are seven proofs required for examination purposes. When this is the case, the proof is covered at the end of the lesson concerned. This gives learners the opportunity to work with the theorem first and gain a better understanding of it before doing the proof.
- These lesson plans do not incorporate the proofs that are not required for examination purposes. This does not mean you should not do the proofs of these theorems with learners – an understanding of the proof can assist learners in making more sense of a theorem.

Breakdown of topic into 7 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision of Gr 8 -10	1,5	5	Cyclic Quadrilaterals	2,5
2	Line/perpendicular from centre	1,5	6	Tangents	3
3	Angle at centre & Angle in a semi-circle	1,5	7	Combination of all theorems and consolidation	2,5
4	Angles in same segment	1			

B SEQUENTIAL TABLE

GRADE 10 and Senior phase	GRADE 11	GRADE 12
LOOKING BACK	**CURRENT**	**LOOKING FORWARD**
• Classifying of 2D shapes • Geometry of straight lines • Geometry of 2D shapes • Theorem of Pythagoras • Similarity and Congruency • Investigate line segments joining the mid-points of two sides of a triangle • Properties of quadrilaterals.	Circle Geometry: • Line drawn from centre of circle and converse • Angle subtended by an arc at the centre of a circle • Angle in a semi-circle • Angles subtended by a chord of the circle • Opposite angles of a cyclic quadrilateral • Exterior angle of a cyclic quadrilateral • Two tangents drawn to a circle from the same point • Tangent perpendicular to radius • The angle between the tangent to a circle and a chord.	• Proportionality theorems • Similar triangles • Theorem of Pythagoras (proof).

C WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are several issues pertaining to Euclidean Geometry.

These include:

- giving incorrect or incomplete reasons
- naming angles incorrectly
- making many irrelevant statements
- not being able to identify cyclic quadrilaterals.

It is important that you keep these issues in mind when teaching this section.

Remind learners that the Euclidean Geometry section requires logical reasoning. There is no short-cut to mastering the skills to answering questions in Euclidean Geometry – it requires continuous and deliberate practice.

ASSESSMENT OF THE TOPIC

D

- CAPS formal assessment requirements for Term 3:
  - Two tests
- Two tests, with memoranda, are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of given information and diagrams related to one or more of the theorems learned. Proofs of some theorems are also required.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

E

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	**Explanation**
Euclidean Geometry	Geometry based on the hypotheses of Euclid. Euclidean geometry deals with space and shape using a system of logical deductions
theorem	A statement that has been proved based on previously established statements
converse	A statement formed by interchanging what is given in a theorem and what is to be proved
rider	A problem of more than usual difficulty added to another on an examination paper
radius	Straight line from the centre to the circumference of a circle or sphere. It is half of the circle's diameter
diameter	Straight line going through the centre of a circle connecting two points on the circumference
chord	Line segment connecting two points on a curve. When the chord passes through the centre of a circle it is called the diameter
circumference	The distance around the edge of a circle (or any curved shape). It is a type of perimeter
segment	The area bound by a chord and an arc
arc	Part of the circumference of a circle
sector	The area bound by two radii and an arc

tangent	Line that intersects with a circle at only one point (the point of tangency)
Point of tangency	The point of intersection between a circle and its tangent line
exterior angle	The angle between any side of a shape, and a line extended from the next side
subtend	The angle made by a line or arc
Theorem of Pythagoras	In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides
hypotenuse	The longest side in a right-angled triangle. It is opposite the right angle
complementary angles	Angles that add up to 90°
supplementary angles	Angles that add up to 180°
vertically opposite angles	Non-adjacent opposite angles formed by intersecting lines
intersecting lines	Lines that cross each other
perpendicular lines	Lines that intersect each other at a right angle
corresponding angles	Angles that sit in the same position on each of the parallel lines in the position where the transversal crosses each line
alternate angles	Angles that lie on different parallel lines and on opposite sides of the transversal
co-interior angles	Angles that lie on different parallel lines and on the same side of the transversal
polygon	A closed 2D shape in which all the sides are made up of line segments. A polygon is given a name depending on the number of sides it has. A circle is not a polygon as although it is a closed 2D shape it is not made up of line segments
quadrilateral	A 4-sided closed shape (polygon)
cyclic quadrilateral	A quadrilateral whose vertices all lie on a single circle. This circle is called the circumcircle or circumscribed circle, and the vertices are said to be concyclic
diagonal	A straight line joining two opposite vertices (corners) of a straight sided shape. It goes from one corner to another but is not an edge
corollary	Theorem that follows on from another theorem

Term 3, Topic 2, Lesson 1

REVISION OF GRADE 8 – 10 GEOMETRY

Suggested lesson duration: 1,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners will have revised: ● geometry concepts from previous years.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have a table ready to complete the Grade 8 theorems as they are discussed as well as four pairs (space permitting) of triangles ready to discuss similarity and congruency.
5. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 9 and 10 textbook or items from a Grade 9 or 10 Geometry test. (Only Via Afrika has a revision exercise).

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. It is important that learners have a good understanding of all the Geometry from previous years. Each theorem that has already been learned will be used often during both Grade 11 and Grade 12 Geometry.
2. Grade 10 Geometry has been included in this lesson (points 26-31), but it may be difficult to work through all of this due to time constraints. Although Grade 10 Geometry is useful in many areas, it does not appear as often as other Geometry in Circle Geometry questions. Use your discretion according to the time available and according to your learners needs.
3. By the end of the lesson, learners should have a detailed summary of the basics of geometry required to manage the Grade 11 work.

DIRECT INSTRUCTION

1. Ask learners what theorems they can remember from Grade 8. Remind them that they learned 10 theorems in Grade 8.
2. Write theorem names on the board as they are given to you.
3. If learners need more clues, tell them that there are nine theorems that are related to finding sizes of angles and one related to finding the lengths of sides.
4. The nine theorems related to finding angles are further split into three regions with three theorems in each.
5. Once learners have told you the theorems they can remember, write this summary on the board and ask learners to write in their exercise books.

ANGLES	Lines	● Vertically opposite angles are equal ● Adjacent angles on a straight line add up to 180° ● Angles around a point add up to 60°
	Triangles	● Angles of a triangle add up to 180° ● The exterior angle of a triangle is equal to the sum of the opposite interior angles ● In an isosceles triangle, the angles subtended by the two equal sides are equal
	Parallel lines	If two parallel lines are cut by a transversal: ● the alternate angles are equal ● the corresponding angles are equal ● the co-interior angles add up to 180°
SIDES	The theorem of Pythagoras states that in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides	

6. It is useful to learn about 'groups' of theorems. For example, if parallel lines are indicated in a question, learners need to go through the three theorems they know pertaining to parallel lines as at least one of them will be used.

7. Ask directed questions that will enable you to ascertain whether any of the Grade 8 theorems need further reinforcement or explanation.

8. Ask: *What do the words congruent and similar mean?* Praise learners who attempt to answer, particularly if their description or definition is accurate.
   Learners should write the following definitions in their books:
   Congruent – For two shapes to be congruent, they must have equal sides and equal angles.
   Similar – For two shapes to be similar, they must have corresponding sides in proportion and their corresponding angles must be equal.

9. Ask: *What are the symbols for similarity and congruency?*
   (Similarity (///) and congruency ($\equiv$))

10. Tell learners: *If you have difficulty remembering these, the congruent sign is like the equal sign ($=$) and congruent means 'exactly equal'.*

11. Remind learners that in this section, the focus is on triangles.

12. When dealing with similar triangles, learners should be able to write up the proportion statements from the statement regarding the triangles being similar.
    Example: If ΔPQR /// ΔTRS then $\frac{PQ}{TR} = \frac{QR}{RS} = \frac{PR}{TS}$

13. Point out that being able to change a similarity statement into a proportion statement is important in Grade 12 Geometry.

14. Tell learners: *To prove two triangles similar one of two things need to be done:*
    - Prove three equal angles
    - Prove that the sides are in proportion.

15. Ask learners again if they have any questions before discussing congruency.

16. Remind learners that congruent triangles are exactly the same size. This means that all six possible measurements (three sides and three angles) are exactly the same.

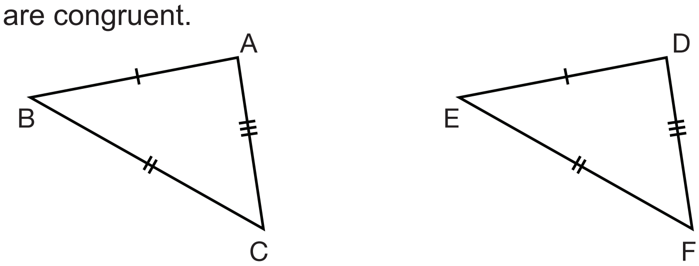
17. Remind learners, that we don't need to find all six measurements equal to prove congruency. In fact, we need only three (but a specific three); because if we find a particular set of three equal measurements, the other three will fall into place and the triangles will be congruent.

18. Ask: *What are the four conditions of congruency?*
On the board, write the abbreviations of any of the conditions for congruency that learners remember. Prompt learners for any conditions for congruency they may have forgotten (SSS, SAS, AAS, RHS).

19. Explain each condition for congruency in detail. Learners must take notes and re-do the diagrams in their books as you describe as follows:

a) **SSS** – side, side, side

If three sides of one triangle are equal in length to three sides of another triangle then the two triangles are congruent.



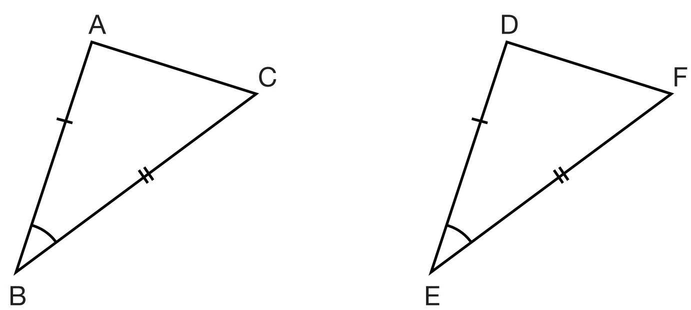
$$\Delta ABC \equiv \Delta DEF$$

Discuss the congruency statement you have just written Point out to learners how important the order is. When written in the above format, the following conclusions can be drawn:

• $\hat{A} = \hat{D}$	• $AB = DE$
• $\hat{B} = \hat{E}$	• $BC = EF$
• $\hat{C} = \hat{F}$	• $AC = DF$

b) **SAS** – side, angle, side

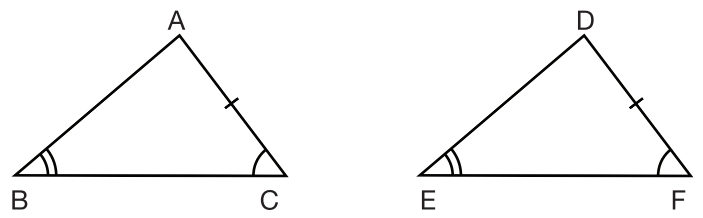
If two sides and the included angle are equal in length or size to two sides and the included angle of another triangle then the two triangles are congruent.



$$\Delta ABC \equiv \Delta DEF$$

c) **AAS** – angle, angle, side

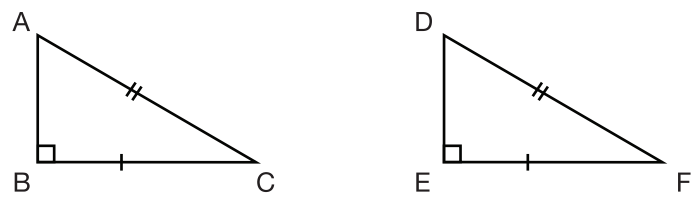
If two angles and a side in one triangle are equal in length or size to two angles and the corresponding side in another triangle then the two triangles are congruent.



$\Delta ABC \equiv \Delta DEF$

d) **RHS** – right, hypotenuse, side

If one side and the hypotenuse in a right-angled triangle are the same length as a side and the hypotenuse in another right-angled triangle then the two triangles are congruent.

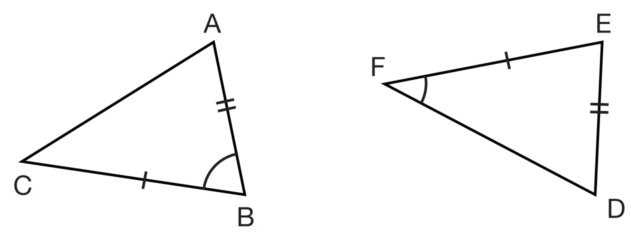


$\Delta ABC \equiv \Delta DEF$

20. Two of these conditions need further explanation as they have extra conditions that need considering before concluding that two triangles may be congruent.

SAS-side, angle, side
If two triangles have two sides of equal length and one angle of the same size they may be congruent. However, the following needs to be checked as well:
Is the angle included?
Does the angle that is marked equal lie between the two sides marked equal?

- The following triangles are <u>not</u> congruent:



- For these triangles to be congruent, angle E needs to be equal in size to angle B, as these are the two angles that lie between the two sides marked equal (the angle is included).

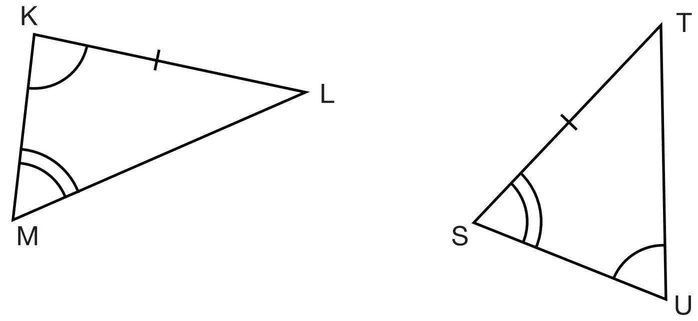
e) **AAS**-angle, angle, side

If two triangles have two angles equal in size and one side the same length they may be congruent. Check the following:

Is the side corresponding in each triangle?

Are the sides that are marked equal sitting in the same place in each of the triangles?
The side should be opposite the same angle in each of the triangles.

- The following triangles are not congruent:



- For these triangles to be congruent, TU needed to be equal to KL (they are both opposite the angle marked with a double arc) OR ML needed to be equal to ST (they are both opposite the angle marked with a single arc).
  Note: KM could have also been marked equal to SU as they are both opposite the angle with no marking.

21. Point out that if they are asked to prove two triangles congruent, the correct order (in other words with the paired sides and angles that are equal) is always given.

22. For example, if the question asks to prove that $\Delta ADE \equiv \Delta PQR$, then there is already evidence as to where to start looking for what sides or angles might be equal. In this case, $AD = PQ, DE = QR$ and $AE = PR$. As well as: $\hat{A} = \hat{P}, \hat{D} = \hat{Q} \ \& \ \hat{E} = \hat{R}$. HOWEVER, make it clear that this is not to say they can use this in their proof. It should merely be used as a guide as to where to look for equal sides or angles if they are stuck.

23. There are some reasons that are used very often in congruency, so it is worth having a summary of these in their books. Discuss the following with learners as they write the summary in their exercise books:

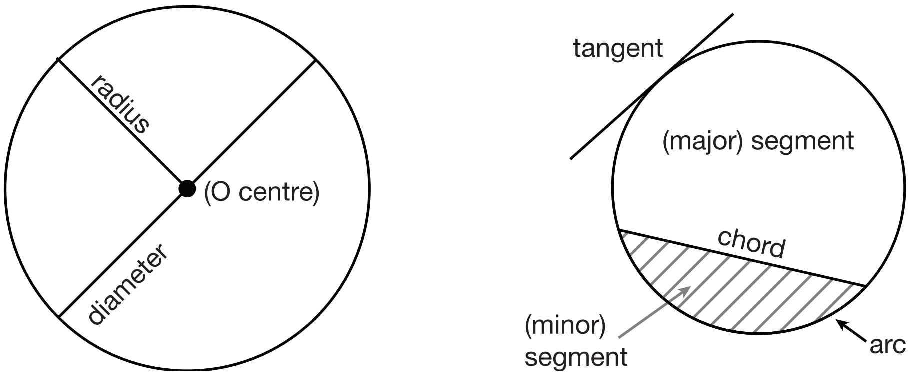
Given	When the information has been given in the question or on the diagram and no actual knowledge is necessary
Common	When a triangle shares a side or angle with another triangle making it the same length or size
Vertically opposite angles equal	This is common when you see the shape of a bowtie
Radii (plural of radius)	This is common when triangles are drawn inside circles – look out for lines drawn from the centre. Remember that all radii are equal in length in a circle

Alternate or corresponding angles	Look for this when any parallel lines are marked anywhere on the diagram

24. You may need to remind learners about the different parts of a circle. In particular, those that will be useful to this year's work.

25. Draw two circles on the board. Ask for volunteers to draw in or mark the following parts (put four parts on one circle and three on the other):
   - centre
   - radius
   - diameter
   - chord
   - segment
   - tangent
   - arc.

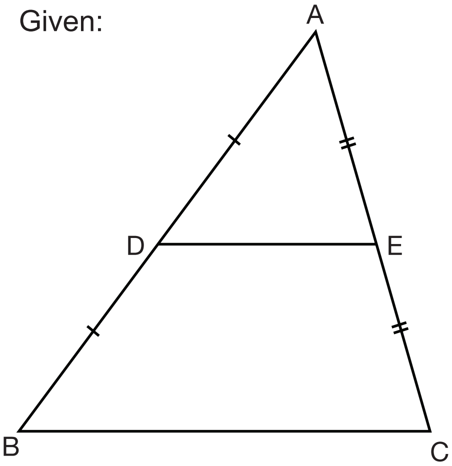
   Ensure learners have the following in their books:



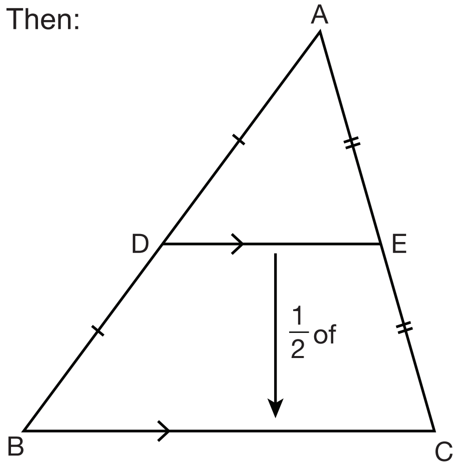
26. Ask: *What can you tell me about the midpoint theorem?* Ask for a volunteer to do the drawing on the board and then discuss what the midpoint theorem tells us according to his/her drawing.

27. Ensure the following is covered for learners to write in their exercise books:

Given:



Then:

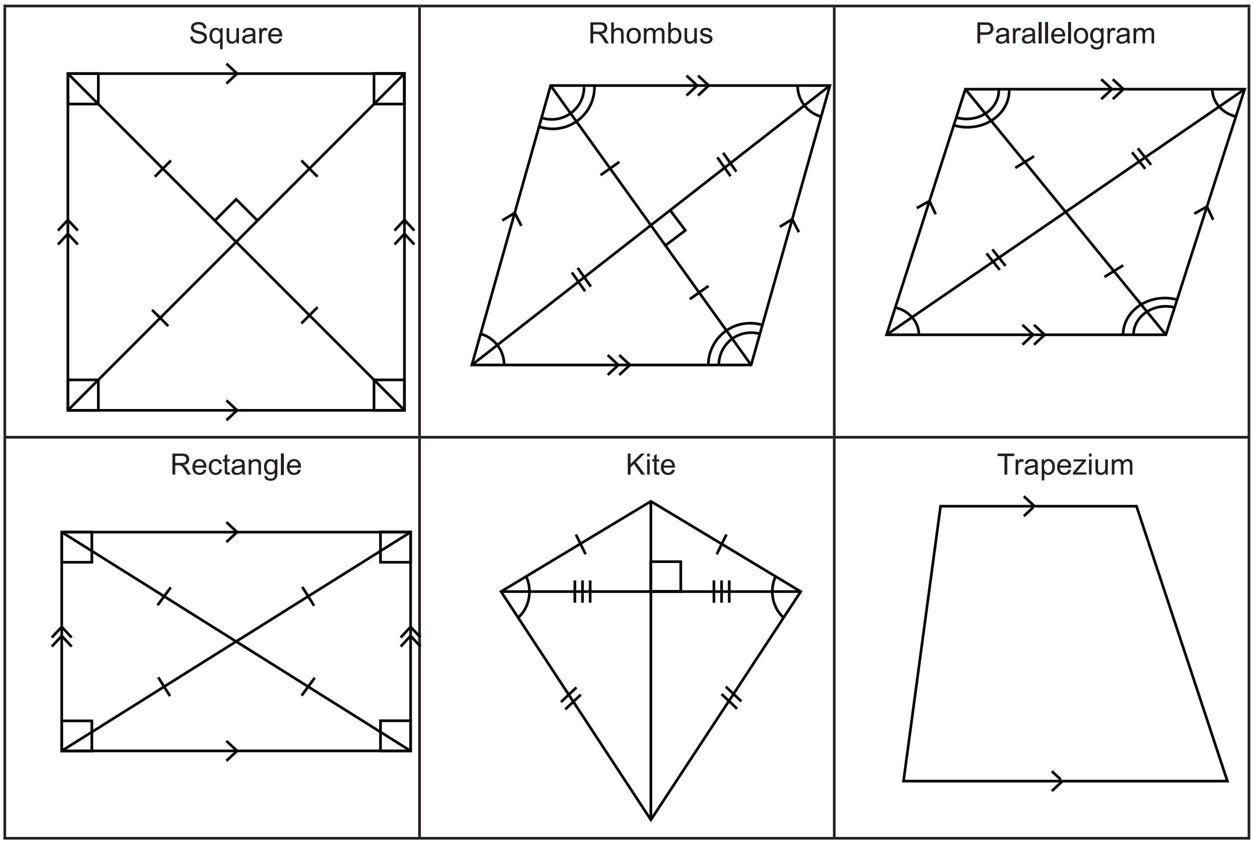


28. Tell learners to write the full theorem: The line which joins the midpoints of two sides of a triangle is parallel to the third side of the triangle and equals half its length.

29. Ask learners what else they covered in Grade 10 (properties of quadrilaterals).

30. Compile a summary of properties by drawing sketches and marking the properties as each shape is discussed.

31. Learners should have a summary like this in their books by the end of the discussion. Encourage learners to give you the properties, and then mark them on the sketch.



32. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

33. Give learners an exercise to complete with a partner (a test from previous years as explained at the beginning of the lesson would be ideal).

34. Walk around the classroom as learners do the exercise. Support learners where necessary.

35. Give learners the following task to do at home before the next lesson:
Draw two circles on a sheet of A4 paper. Ensure that the centre is marked clearly.
If it is possible to draw them in pen, that would be better.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=uZYjnVp8IXo
(Angle and line theorems)

https://www.youtube.com/watch?v=OEp7YK6WEXE
(Congruency and similarity)

Term 3, Topic 2, Lesson 2

LINE/PERPENDICULAR FROM CENTRE

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners should be able to: • prove the theorem, 'the line drawn from the centre of a circle perpendicular to the chord bisects the chord' • answer riders on the theorems, the line drawn from the centre of a circle perpendicular to the chord, the line drawn from the centre of a circle to the midpoint of a chord and the perpendicular bisector of a chord.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 3 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw two circles with the centre marked.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	214	1	192	1	209	9.1	278	8.1	339

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

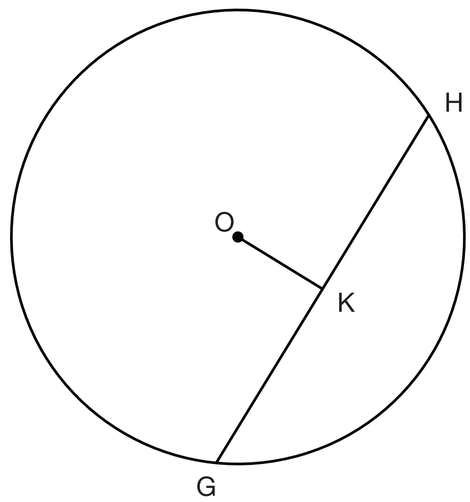
1. Ensure learners have their two circles ready – these will be used for learners to measure and understand the theorem for themselves.
2. While giving the instructions to learners to draw, measure or mark off on their own circle, walk around the class to assist where necessary. Try not to stand at the front of the class.
3. Learners should work in pencil, so they can erase after the lesson and then use the circles in all the lessons that follow while learning new theorems.

DIRECT INSTRUCTION

1. Ask learners to
   a) Draw in a chord anywhere on one of their circles and label it GH.
   b) Draw a line from the centre to the chord GH, ensuring the angle formed at the chord with the line is 90°.

   (If learners don't have a protractor, use the corner of a page to ensure it is a right angle). Label the point K.
2. Tell learners to measure their chord GH (and write the measurement on the line) as well as the length of GK and KH. Ask what they notice. If learners don't notice anything, tell them to turn to a partner and look at their drawing and measurements to see if anything is noticeable.

3. Draw your own sketch on the board and ask learners what they have noticed. The line from the centre should have bisected the chord. Ensure you use the word 'bisect' instead of 'cut in half'. Learners need to be familiar with the correct vocabulary.



4. Point out that it didn't matter where they drew their chord or how close it was to the centre or the circumference. They could check now by looking at how different some other learners' diagrams looked, and observing that the same conclusion was reached.

5. Ask: *What is the converse of a theorem?*
   (A statement formed by interchanging what is given in a theorem with what is to be proved).

6. Ask*: What could the converse of the theorem you have just learned be?*
   (If a line is drawn from the centre of a circle to the midpoint of a chord, it will be perpendicular to the chord).

7. Tell learners we will show that now by using the second circle that they have ready.

8. Tell learners to:
   a) Draw in a chord anywhere on one of their circles and label it PQ.
   b) Measure the length of the chord PQ and mark the centre point, calling it R.
   c) Draw a line from the centre to point R.
   d) Measure the angle at R.

9. Learners should find that the angle at R is 90°.

10. Ask learners to write the theorem and its converse in full into their books as well as the acceptable abbreviated form that can be used when using the theorem in a question. Tell learners to copy the following table into their books and add an extra row at the bottom which will be filled later in the lesson.

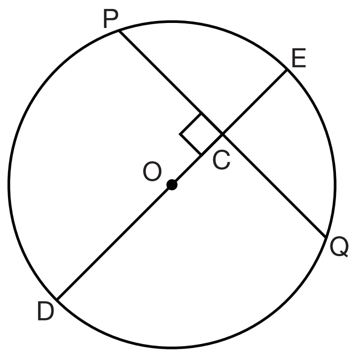
TOPIC 2, LESSON 2: LINE/PERPENDICULAR FROM CENTRE

Theorem	Acceptable abbreviated form
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	Line from centre ⊥ to chord
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	Line from centre to midpt chord

11. Ask learners to highlight this summary in a way that it will be easy for them to find later when many theorems have been covered.

12. Do two worked examples with learners. Learners should write the worked examples in their books.

> The diagrams used in the worked examples are available in the Resource Pack. (Resource 3).

Example 1	Teaching notes
In the diagram, O is the centre of the circle. The diameter DE is perpendicular to the chord PQ at C. DE = 20cm and CE = 2cm.  Calculate the lengths of the following with reasons: a) OC	Tell learners that they always need to read through the information carefully and check it on the diagram. This check should cover two things: 1. *Does the information make sense when the diagram is considered?* 2. *Is all the information displayed on the diagram?* – if not transfer it. In this case, ask: What information can be added to the diagram? (DE = 20cm and CE = 2cm) *Ask: What can you say from the information and diagram that might help with the answer?* (DE is a diameter; therefore, OD and OE are radii and must be 10cm each. As CE is given this also means that OC is simple to work out.)

b) PQ EXEMPLAR 2013	Ask learners for ideas on how to find PQ. If they are struggling, nudge them in the right direction rather than show them the solution. Tell learners they know the length of the radius. Ask: *Where else can the radius be seen on the diagram?* (OQ and OP). If they joined OQ and OP what do they notice (a right-angled triangle) Ask: *What would you find if you worked in one of those right-angled triangles?* (PC or CQ) Ask: *How would that help?* (Perpendicular from centre bisects chord therefore the other one equals the same).

Solution:

a) $\mathrm{CE} = 2\mathrm{cm}$

$\mathrm{OE} = 10\mathrm{cm}$ (O is midpoint of DE)

$\mathrm{OC} = \mathrm{OE} - \mathrm{CE}$

$\therefore \mathrm{OC} = 8\mathrm{cm}$

b) In $\triangle \mathrm{COQ}$:

$\mathrm{CQ}^2 + \mathrm{OC}^2 = \mathrm{OQ}^2$

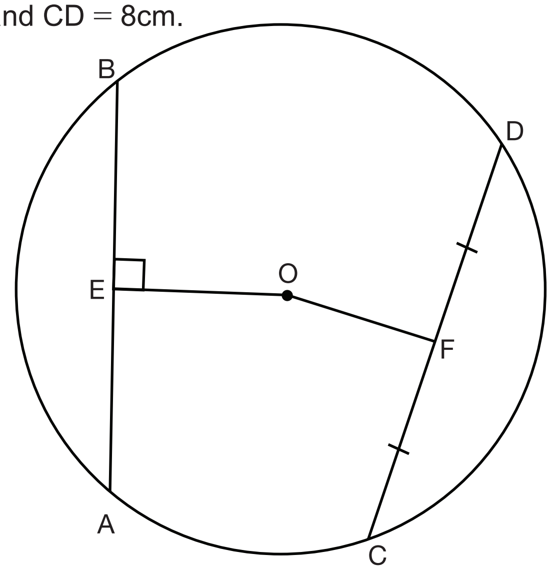
$\mathrm{CQ}^2 + (8)^2 = (10)^2$ (Pythagoras)

$\mathrm{CQ}^2 = 36$

$\mathrm{CQ} = 6$

$\mathrm{PC} = \mathrm{CQ}$ (line from centre $\perp$chord)

$\therefore \mathrm{PQ} = 12\,\mathrm{cm}$

Example 2	**Teaching notes**
In the figure below, AB and CD are chords of the circle with centre O. OE ⊥ AB, CF = FD, OE = 4cm, OF = 3cm and CD = 8cm. 	Tell learners to read through the information carefully and check it with the diagram. Ask: *What given information can you add to the diagram?* (Length of OE, OF, CF and FD)
a) Calculate the length of OD.	Once all measurements have been filled in on the diagram, learners should see that this is a straightforward theorem of Pythagoras question.
b) Hence, calculate the length of AB. EC 2015	Ask: *What else can you fill in on the diagram once OD has been found?* (OC, OA and OB as they are all radii to the circle). Ask: *How will this help you in finding AB?* (Use of the theorem of Pythagoras again as well as the theorem, the line drawn from the centre of a circle perpendicular to a chord bisects the chord).

Solution:

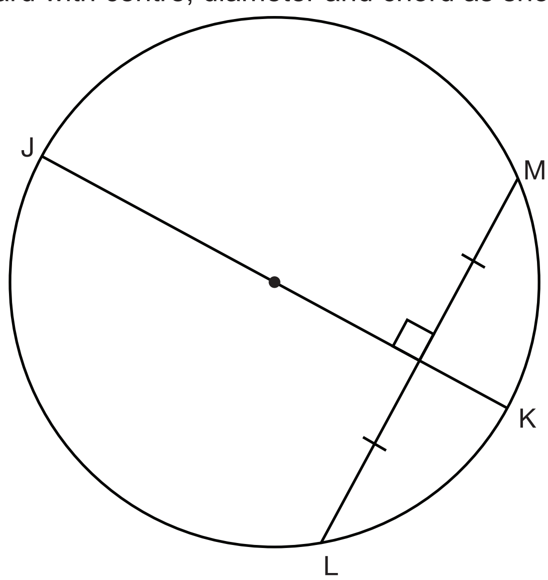
a) In $\triangle ODF$:

$$OD^2 = OF^2 + DF^2 \text{ (Pythagoras)}$$
$$OD^2 = (3)^2 + (4)^2$$
$$OD^2 = 25$$
$$\therefore OD = 5\text{cm}$$

b) In $\triangle OBE$:

$$BE^2 = OE^2 + OB^2$$
$$BE^2 = (4)^2 + (5)^2$$
$$BE^2 = 9 \quad \text{(Pythagoras)}$$
$$BE = 3$$
$$BE = AE \quad \text{(line from centre } \perp \text{ chord)}$$
$$\therefore AB = 6\text{cm}$$

13. Tell learners: *There is one more theorem to be covered in this lesson. It has a connection to the previous theorem and its converse.*
14. Draw a circle on the board with centre, diameter and chord as shown below.



15. Use the diagram to show that if a line (JK) passes through any chord (ML) and is the perpendicular bisector of that chord (crosses at a right angle and cuts the chord in half), then that line passes through the centre of the circle.
16. Tell learners that you will look at this theorem again at the end of the lesson when you prove that it is true.
17. Ask learners to add the theorem and its abbreviated form to the bottom of the table they worked on earlier.

The perpendicular bisector of a chord passes through the centre of the circle.	Perp bisector of chord

18. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

19. Give learners an exercise (from their textbook or any other suitable exercise) to complete on their own.

20. Walk around the classroom as learners do the exercise. Support learners where necessary.

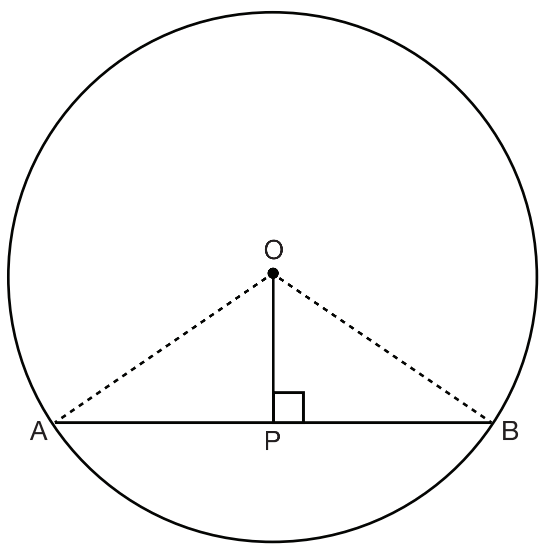
21. Tell learners that now you are going to prove the theorem that they have just learned and practiced. Explain that all theorems can be proved using other theorems. Proving why a theorem works can often offer an understanding of another aspect of Geometry. Understanding why a theorem will always work may also assist learners in remembering the theorem better. Tell learners that when proving a theorem, it is accepted practice to use other previously accepted statements (other theorems) but not the statement regarding the theorem that is being proved.

22. Point out that using the theorem and proving the theorem are very different. In the exercises that they have completed so far, learners used the theorem. Now they will prove it.

23. Go through the proof of 'The perpendicular drawn from the centre of a circle to a chord bisects the chord'. As it is the first theorem you are proving with them, take the time to explain how the same headings are always used: given, required to prove (RTP) and proof.

24. Ask learners to write the proof of the theorem at the back of their exercise books so that all the proofs they need to learn for the exams are in one place. The heading should be the theorem written in full.

Proof for: The perpendicular drawn from the centre of a circle to a chord bisects the chord.



GIVEN: Circle with centre O and chord AB. OP $\perp$ AB.

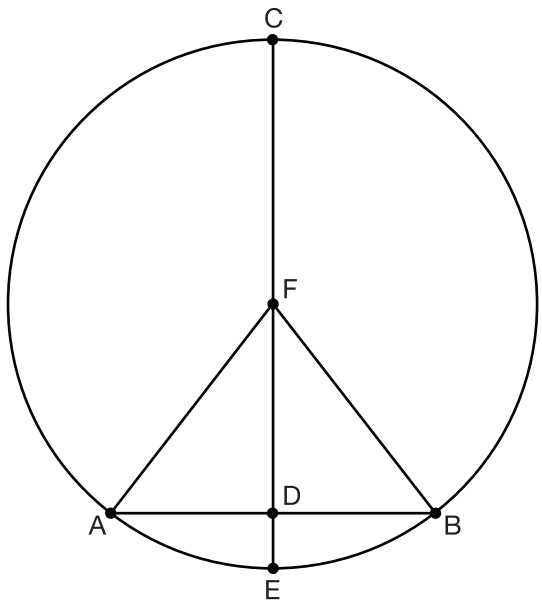
RTP: AP = PB

PROOF:

Statement	**Reason**
Join OA and OB	
In ΔAOP and ΔBOP	
$OA = OB$	Radii
$OP = OP$	Common
$A\hat{P}O = B\hat{P}O = 90^{\circ}$	Given
$\therefore \Delta AOP \equiv \Delta BOP$	RHS
$\therefore AP = PB$	

25. Learners are not required to know the proofs of the converses. Tell learners, therefore, that we will now go back to the other theorem mentioned earlier.

<u>Proof for: The perpendicular bisector of a chord passes through the centre of the circle.</u>



GIVEN: A circle with chord AB. AD = DB, AB $\perp$ CE.

RTP: F is the centre of the circle

PROOF:

Statement	**Reason**
In ΔADF and ΔBDF	
$AD = DB$	Given
$DF = DF$	Common
$A\hat{P}O = B\hat{P}O = 90^{\circ}$	Given
$\therefore \Delta ADF \equiv \Delta BDF$	SAS
$\therefore AF = BF$	
$\therefore$ F is the centre	Any point equidistant from 2 points on the circumference is the centre of the circle.

26. Ask learners if they have any questions.
27. Remind learners that they will need their drawn circles again for the next lesson. They will also need a protractor –one per two learners will be acceptable.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=LOAe8vbxbp0&t=37s
(Proof of theorem: The perpendicular drawn from the centre of a circle to a chord bisects the chord)

https://www.youtube.com/watch?v=36B3hxQRI1o
(Proof of perpendicular bisector of chord passes through circle)

Term 3, Topic 2, Lesson 3

ANGLE AT CENTRE AND ANGLE IN A SEMI-CIRCLE

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners should be able to: ● prove the 'angle at centre' theorem ● answer riders on the theorems, angle at centre and angle in a semi-circle.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resources 4 and 5 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw two circles with the centre marked and two lines labelled AB and FG.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
2 3	216 218	2*	196	2	211	9.2 9.3	283 286	8.2	343

***** This exercise incorporates questions covering angles in the same segment which is only covered in the next lesson.

CONCEPTUAL DEVELOPMENT

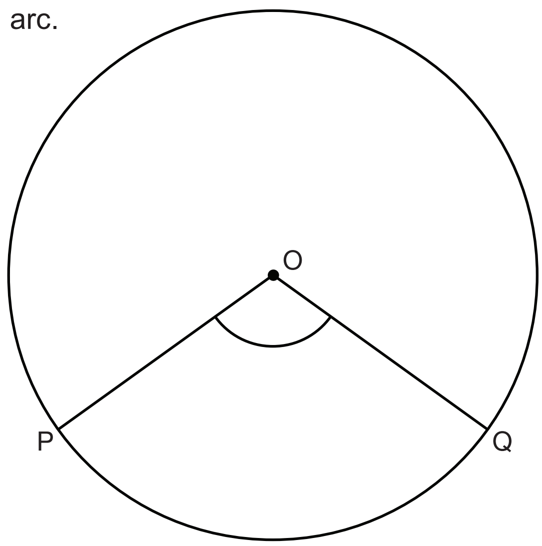
C

INTRODUCTION

1. Ensure learners have their two circles ready – these will be used for learners to measure and understand the theorem for themselves.
2. While giving the instructions to learners to draw, measure or mark off on their own circle, walk around the class to assist where necessary.
3. Learners should work in pencil, so they can erase after the lesson then use the circles in all the lessons that follow while learning new theorems.

DIRECT INSTRUCTION

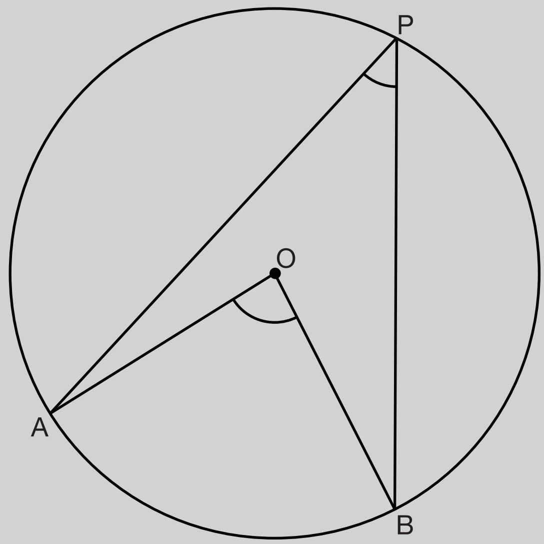
1. Start the lesson by explaining the word 'subtends' to learners. Subtends is an important concept used regularly in most of the circle theorems.
2. Use line segment AB already on the board to form an angle being subtended from it. Call this angle C. Use both hands to point at A and at B then show how, by running your fingers along the lines you have just added, your fingers come together at the angle subtended from AB (C). Ask for a few volunteers to:
   a) Subtend another angle from AB but still going in the same direction
   b) Subtend another angle from AB but going in the opposite direction
   c) Subtend a few angles from FG going in both directions.
3. Use the diagrams to make a few statements and to use the correct terminology.
   For example, C is subtended from AB or AB subtends C.
4. Show that an angle can also be subtended from an arc.



$\hat{O}$ is substended from arc PQ. PQ subtends $\hat{O}$.

5. Tell learners that this idea is going to be used now to investigate the next theorem.

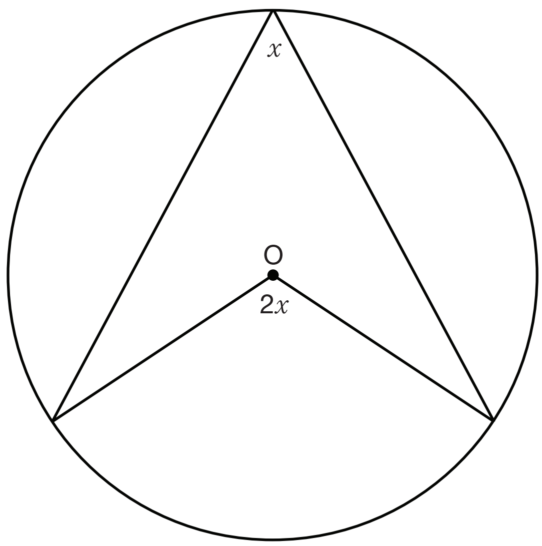
Note: Do the following investigation with the learners to ensure that they encounter this theorem in its most common form (the arrowhead) first, before showing them the other two versions.



6. Ask learners to
   a) Mark two points, A and B on the circumference.
   b) Subtend an angle from the arc AB to the centre.
   c) Subtend an angle from the arc AB to the circumference.
   d) Measure the angle at the centre.
   e) Measure the angle at the circumference.

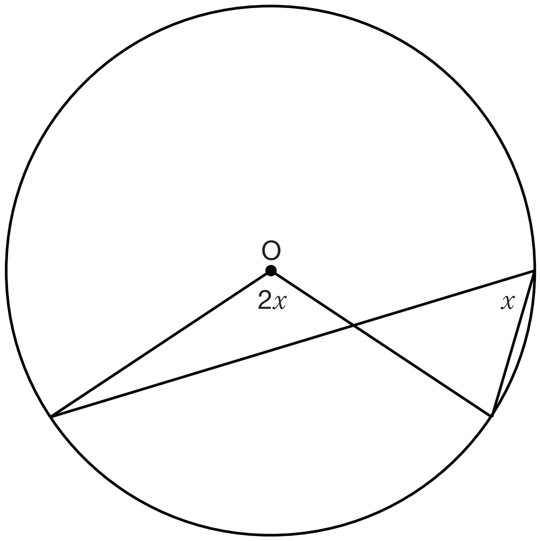
   Ask: *What do you notice? Compare what you notice on your diagram with the person sitting next to you.*
   (The angle at the centre is double the size of the angle at the circumference).

7. Use your own diagram. Mark the angle at the centre $2x$ and the angle at the circumference x.



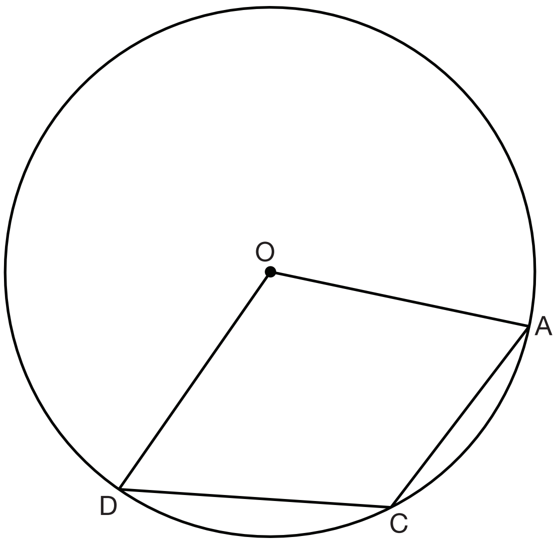
8. Tell learners that the angle at the circumference does not always have to be above the angle at the centre and therefore forming the 'arrowhead' formation.

9. Mark two points on the second circle you have on the board. Subtend an angle at the centre from the arc then subtend one at the circumference from the arc. This time, subtend it to the side of the centre.



10. Ensure learners can see that this shows the theorem too – the angle at the centre is subtended from the same arc that subtends the angle at the circumference.

11. Show the third version of the theorem. Mark two points on the circumference. Subtend an angle at the centre from the arc. Subtend an angle at the circumference from the same arc but in the opposite direction.



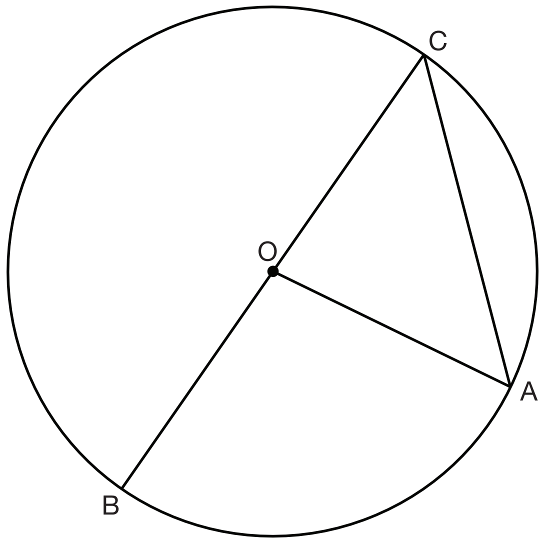
Point out to learners that when the angle at the circumference is subtended in the opposite direction to the angle at the centre, then the reflex angle at the centre is the angle of interest.

In other words, reflex $\text{B}\hat{\text{O}}\text{A} = 2 \times \text{B}\hat{\text{C}}\text{A}$

Mark the reflex angle at the centre $2x$ and the angle at the circumference x now.

12. Learners should sketch all three versions of the same theorem, with the angles marked $2x$ and x accordingly, in their exercise books.

13. Show learners one more diagram that also represents this theorem but sometimes confuses learners.



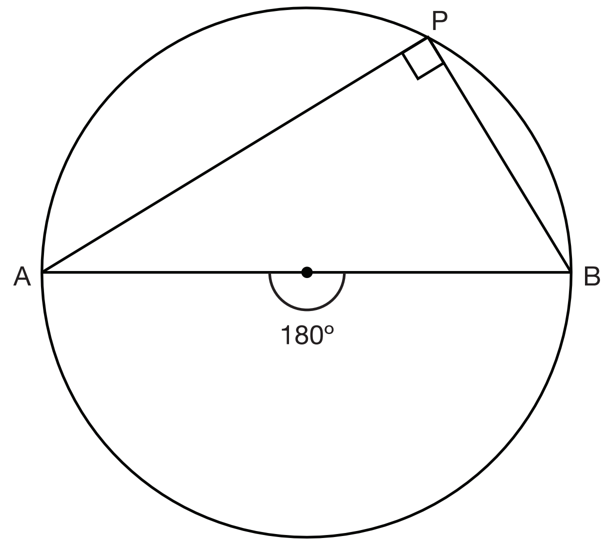
Tell learners to note again (by showing them on the board) that the angle at the centre $(\mathrm{B\hat{O}A})$ is subtended from arc AB and the angle at the circumference $(\mathrm{B\hat{C}A})$ is also subtended from arc AB. Therefore, $\mathrm{B\hat{O}A} = 2 \times \mathrm{B\hat{C}A}$

14. Ask learners to write the theorem in full into their books as well as the acceptable abbreviated form that can be used when using the theorem in a question. Tell learners to copy the following table into their books. Note that they should add an extra two rows at the bottom. These rows will be filled later in the lesson.

Theorem	**Acceptable abbreviated form**
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle.	Angle at centre $= 2 \times <$ at circumference.

15. Look at a special case of this theorem. Ask learners to erase the pencil from one of their circles then:
    a) Draw in diameter AB.
    b) Regard AB as the arc on a circumference and use it to subtend an angle at the circumference.
    Ask learners the measure of the angle at the centre (180°). Ask: *Using the theorem you have just learned, what do you expect the angle at the centre to be?* (90°).

16. Draw this diagram on the board:



17. Ask learners to write this theorem on the next row of their table.

The angle subtended by the diameter at the circumference of the circle is 90°	< in semi-circle

18. Ask: *What do you think the converse of this theorem would be?*
(If the angle at the circumference is 90°, then the chord it is subtended from must be a diameter)

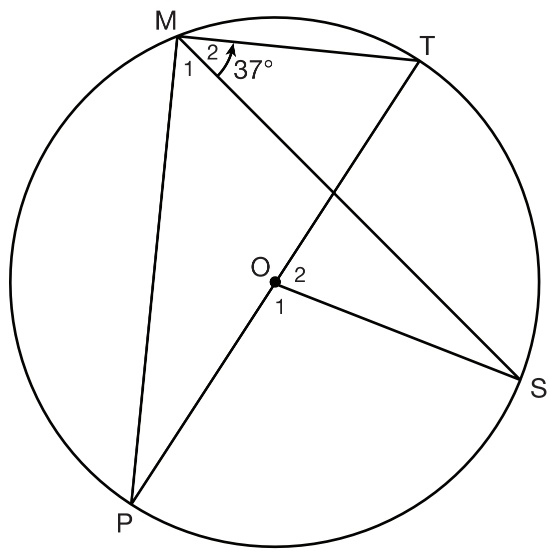
19. Tell learners to write this converse in the last line of the table.

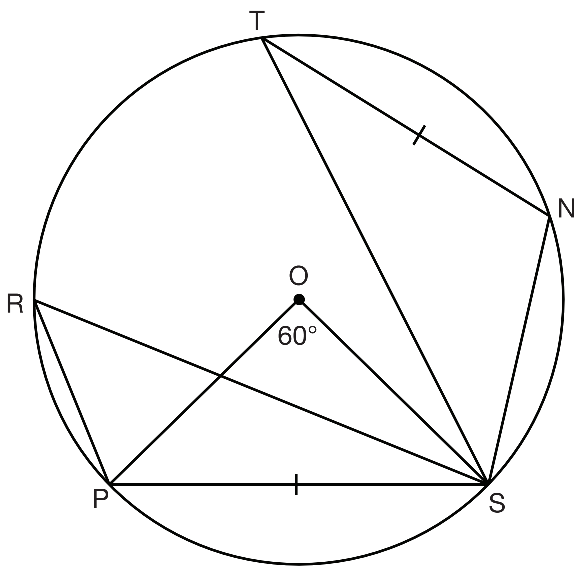
If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter.	Chord subtends 90°

20. Ask learners to highlight this summary in a way that it will be easy for them to find later when many theorems have been covered.

21. Do two worked examples with learners. Learners should write the worked examples in their exercise books.

The diagrams used in the worked example are available in the Resource Pack. (Resource 4).

Example 1	**Teaching notes**
In the diagram below, PT is a diameter of the circle with centre O. M and S are points on the circle on either side of PT. MP, MT, MS and OS are drawn. $\hat{M}_2 = 37^o$  Calculate, with reasons, the size of: a) $\hat{M}_1$ b) $\hat{O}_1$ NSC NOV 2014	Tell learners to read through the information carefully and check it with the diagram. Ask: *What information can you add to the diagram?* Even though everything mentioned in the question is shown on the diagram, point out that the words 'centre' and 'diameter' tell them something useful. Ask: *What do you know about* $P\hat{M}T$ (It equals 90°). Tell learners to mark it as such. Advice: Learners must remember what they marked and were not given so that when they use the information in the solution they must remember to give a reason for knowing the information.
Solution: a) $P\hat{M}T = 90^o$ $\therefore \hat{M}_1 = 53^o$	< in semi-circle
b) $\hat{O}_1 = 106^o$	< at centre $= 2 \times <$ at circumference

<table>
<tr><th>Example 2</th><th>Teaching notes</th></tr>
<tr><td>O is the centre of the circle TNSPR.
$P\hat{O}S = 60^o$ and $PS = NT$.


Calculate the size of $P\hat{R}S$</td><td>Note: This example comes from an exam. There was a second part of the question which relates to a theorem not covered yet. Discuss how to find $P\hat{R}S$ and what theorem will be used. Once the solution has been written up, ask learners to note that there are two chords marked equal in length in the diagram. Tell them you will come back to this diagram in the following lesson to discuss what rules apply.</td></tr>
<tr><td>Solution:
$P\hat{R}S = 30^o$</td><td>< at centre $= 2 \times <$ at circumference</td></tr>
</table>

22. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

23. Give learners an exercise to complete on their own.

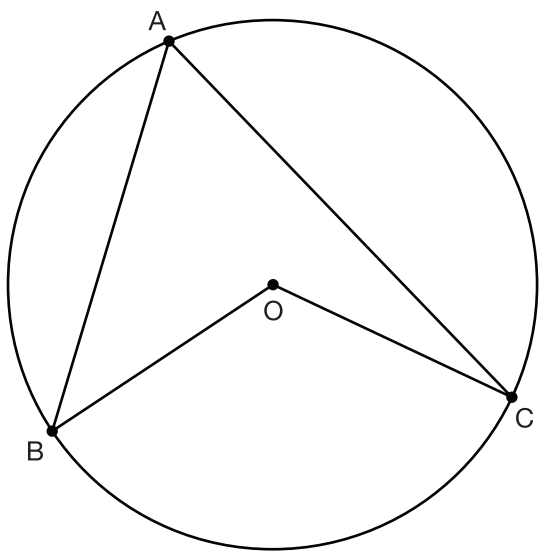
24. Walk around the classroom as learners do the exercise. Support learners where necessary.

25. Tell learners that you are going to prove the theorem that they have just learned and practiced.

26. Ask learners to write the proof in the back of their books so that all the proofs they need to learn for the exams are in one place. The heading should be the theorem written in full.

Proof for: The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle.

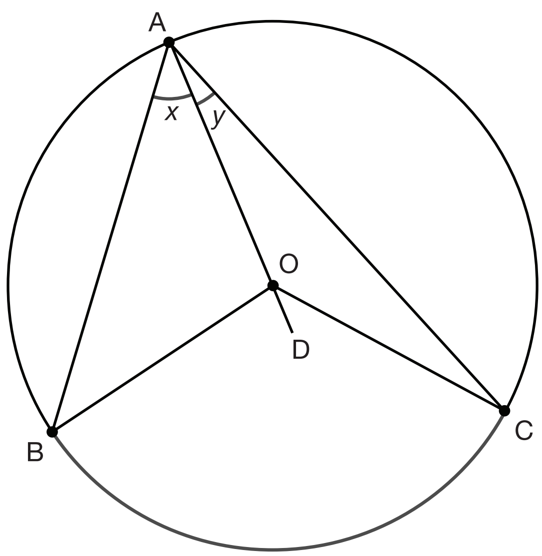
Start with this basic representation of the theorem:



GIVEN: Circle with centre O and A, B and C are all points on the circumference.

RTP: $\hat{BOC} = 2 \times \hat{BAC}$

PROOF: Join AO and produce to D. Let $\hat{BAO} = x$ and $\hat{CAO} = y$



Statement	Reason
$\text{A}\hat{\text{B}}\text{D} = x$ and $\text{A}\hat{\text{C}}\text{D} = y$	radii ($\text{AO} = \text{BO}$ and $\text{AO} = \text{OC}$)
$\text{B}\hat{\text{O}}\text{D} = 2x$	ext < of ΔABO
$\text{C}\hat{\text{O}}\text{D} = 2y$	ext < of ΔACO
$\therefore \text{B}\hat{\text{O}}\text{D} + \text{C}\hat{\text{O}}\text{D} = 2x + 2y$	
$\therefore \text{B}\hat{\text{O}}\text{C} = 2(x+y)$	
$\therefore \text{B}\hat{\text{O}}\text{C} = 2 \times \text{B}\hat{\text{A}}\text{C}$	

27. CAPS does not require that learners know the proof of the angle in a semi-circle.

28. Ask learners if they have any questions.

29. Remind learners that they will need their drawn circles again for the next lesson. They will also need a protractor –one per two learners will be acceptable.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=y7-yT5qUtN0
(Angle at centre proof)

Term 3, Topic 2, Lesson 4

ANGLES IN SAME SEGMENT

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners should be able to: • prove the theorem • answer riders on the theorem, angles in the same segment.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resources 5 to 9 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw two circles.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4 5	220 221	2	196	3	214	9.4	293	8.3	347

CONCEPTUAL DEVELOPMENT

C

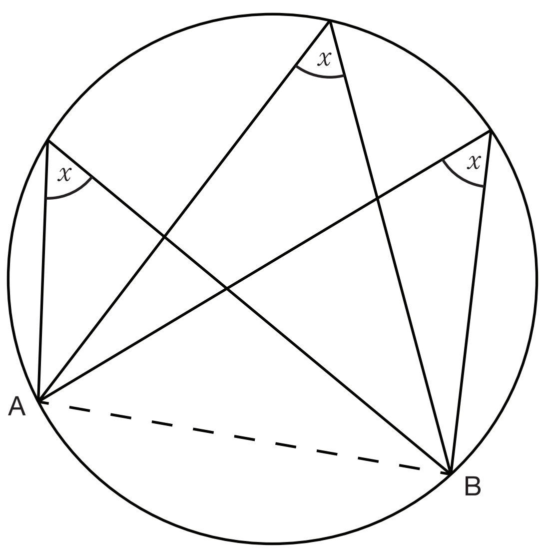
INTRODUCTION

1. Ensure learners have their two circles ready – these will be used for learners to measure and understand the theorem for themselves.
2. While giving the instructions to learners to draw, measure or mark off on their own circle, walk around the class to assist where necessary. Avoid standing at the front of the class.
3. Learners should work in pencil, so they can erase after the lesson then use the circles in all the lessons that follow while learning new theorems.

DIRECT INSTRUCTION

1. Ask learners to:
   a) Mark two points, A and B on the circumference.
   b) Subtend three angles from the arc AB to the circumference.
   c) Measure all three angles.

   Ask: *What do you notice about the angles?* (All the angles are equal).
2. Mark two points A and B on one of the circles on the board. Draw in a dotted line to show the chord. Subtend three angles into the same segment and mark them all .



3. Tell learners that no matter how many angles are subtended from the same arc they will always be equal in size. Ensure that learners understand that those angles must always be subtended into the same segment.

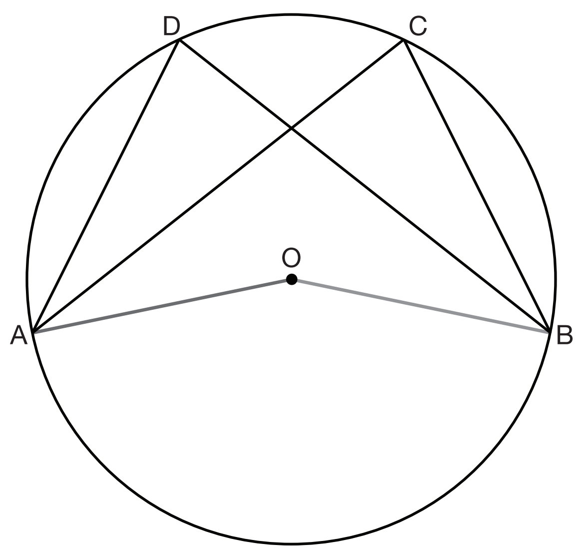
4. Remind learners how a chord splits a circle into two segments. Shade the above diagram to show the two separate segments formed by the chord. Point out that all three angles are being subtended into the same segment (in this case the larger of the two segments).

5. Ask learners to write the theorem in full, as well as the acceptable abbreviated form that can be used when using the theorem in a question, into their books. Tell learners to copy the following table into their books. Note that they should add an extra five rows to the table. These rows will be filled in as the lesson proceeds.

Theorem	**Acceptable abbreviated form**
Angles subtended by a chord of the circle, on the same side of a chord, are equal	<'s in same segment

6. Ask: *What connection can you see between this theorem and the theorem covered in the last lesson?* (Angle at centre equals two times angle at circumference).
Use a diagram to demonstrate:

This diagram is available in the Resource Pack. (Resource 5) It has been enlarged for your convenience.

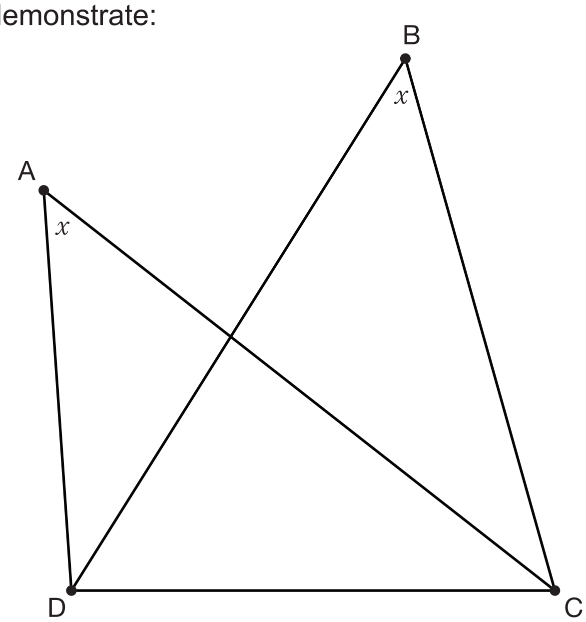


Ask:
What is known about $A\hat{O}B$ *and* $A\hat{D}B$*? (* $A\hat{O}B$ is double the size of $A\hat{D}B$)
What is known about $A\hat{O}B$ *and* $A\hat{C}B$*?* ($A\hat{O}B$ is double the size of $A\hat{C}B$)
What does this tell us about $A\hat{D}B$ *and* $A\hat{C}B$*?* (they must be equal)
What arc subtends these two angles? (AB)
What conclusion can be drawn? (Angles subtended from the same arc (chord) are equal)

This is how the theorem will be proved later in the lesson.

7. The converse of this theorem will make more sense to learners after cyclic quadrilaterals have been covered. Nevertheless, discuss the converse with them now.
   Ask: *What do you think the converse statement would be?*
   (If one line subtends two equal angles in the same direction then the points made by that line and the vertices of the angles will all lie on the circumference of a circle).

8. Use this diagram to demonstrate:



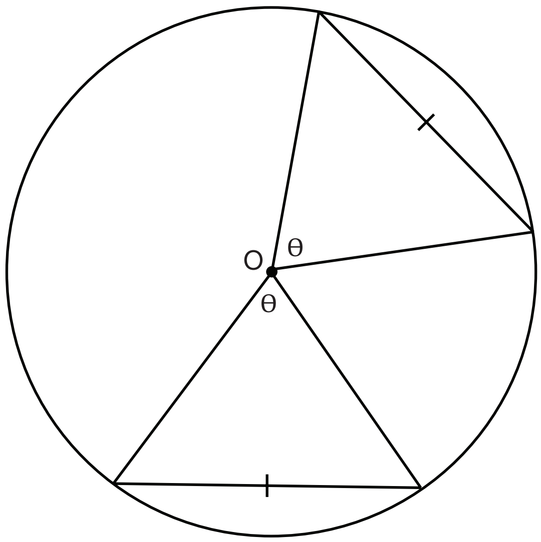
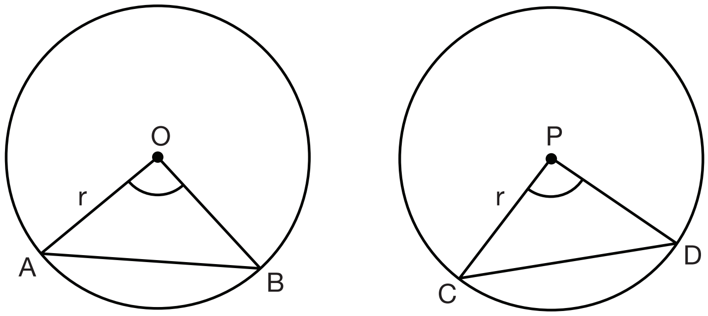
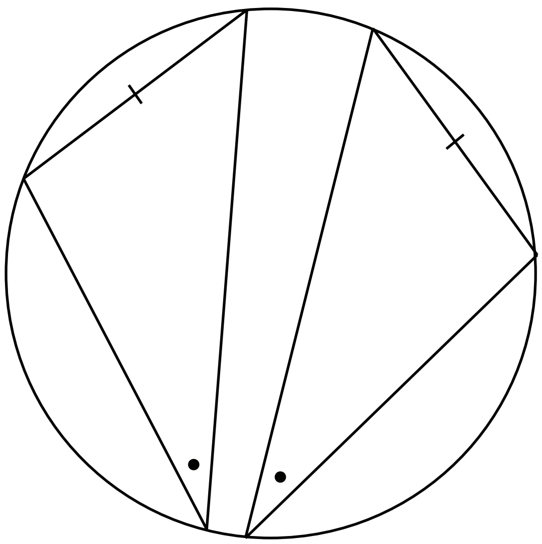
   Show learners that both angles are subtended from the same line (CD) and are on the same side of it. These angles are marked equal. Therefore, the points A, B, C and D are points on a circle. The correct term for this is, the points are concyclic.

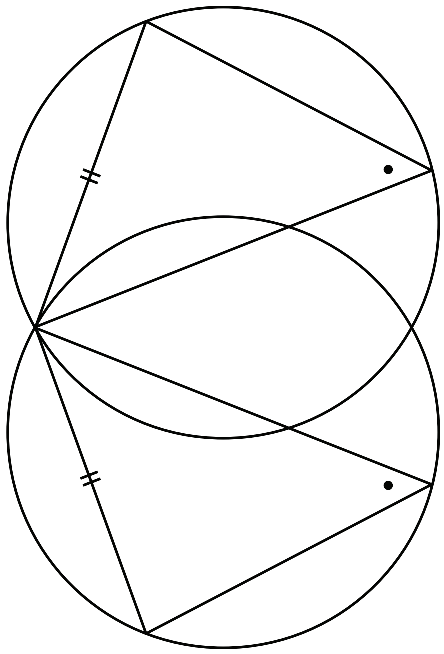
9. Tell learners to add this theorem to the table made earlier.

If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	converse <'s in same segment

10. Discuss the following four diagrams with learners. All diagrams represent corollaries of the theorem, angles in the same segment are equal. A corollary is a theorem that follows on from another theorem.

The following diagrams are available in the Resource Pack. They have been enlarged for your convenience.

Diagram	Discussion
	If two chords are the same length, then the angles subtended by them to the centre will be equal in size.
	If two circles are the same size (equal radii), and two chords in each circle are equal in length, then those chords will subtend equal angles at the centre.
	If two chords are the same length, then the angles subtended by them to the circumference will be equal in size.

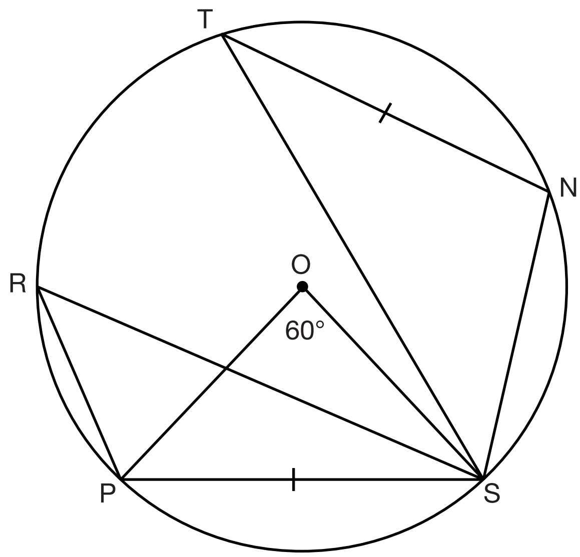
	If two circles are the same size (equal radii), and two chords in each circle are equal in length, then those chords will subtend equal angles at the circumference

11. Once learners have been shown the corollaries visually, ask them to write them into their table with the accepted abbreviated form.

Equal chords subtend equal angles at the circumference of a circle.	Equal chords, equal angles
Equal chords subtend equal angles at the centre of a circle.	Equal chords, equal angles
Equal chords in equal circles subtend equal angles at the circumference of the circles.	Equal circles, equal chords, equal angles
Equal chords in equal circles subtend equal angles at the centre of the circles.	Equal circles, equal chords, equal angles

12. Ask learners to highlight this summary in a way that it will be easy for them to find later when many theorems have been covered.

13. Point out that these theorems are not used as often as the others but are still worth knowing. Ask learners to turn back to the second example you did in Lesson 3:



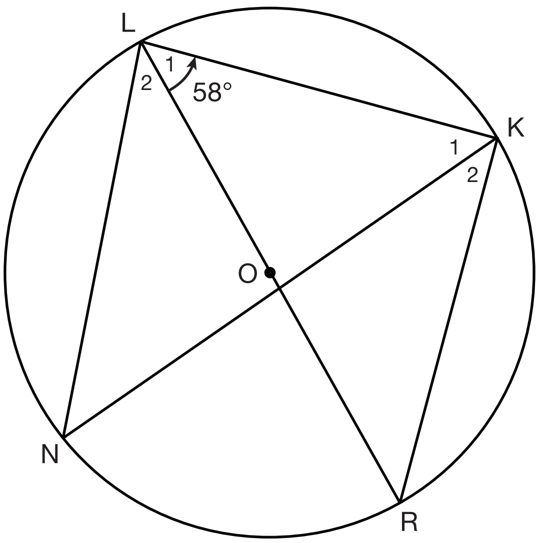
This question had a second part to it asking for the size of $T\hat{S}N$.
Ask: *What is the solution?*
(30°)
It is the same size as $P\hat{R}S$ because equal chords subtend equal angles at the circumference of a circle.

14. Do two worked examples with learners. Learners should write the worked examples in their exercise books.

The diagrams used in the worked example are available in the Resource Pack. Resource 9.

Example 1

In the diagram, O is the centre of the circle. Diameter LR subtends $L\hat{K}R$ at the circumference of the circle. N is another point on the circumference and chords LN and KN are drawn. $\hat{L}_1 = 58^o$



Calculate, giving reasons, the size of:

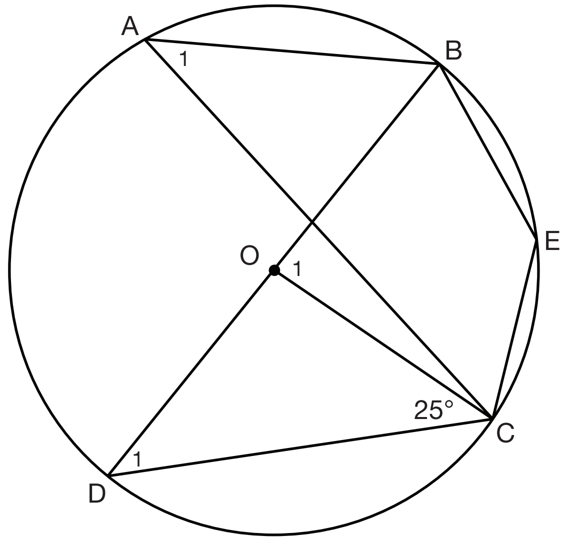
a) $L\hat{K}R$
b) $\hat{R}$
c) $\hat{N}$

NSC NOV 2015

Teaching notes

Point out that questions will become a little more complex as their knowledge of the different theorems builds up.
This may be seen by some as a disadvantage. Remind learners that there are often more opportunities of getting the solution when many different theorems can be used.
Ask: *Is there is any information in the question that should be transferred onto the diagram or anything you can fill in that you already know*?
($N\hat{L}K$ and $R\hat{K}L$ are both 90° as they are in a semi-circle).
Remind learners that they could be required to use any theorem they have ever learned at any time – including theorems from Grade 8.

Solution: a) $L\hat{K}R = 90^o$ b) $\hat{R} = 32^o$ c) $\hat{N} = 32^o$	 < in semi-circle sum of <'s in Δ <'s in same segment

Example 2	Teaching notes
In the figure below, $D\hat{C}O = 25^o$ and O is the centre of the circle. A, B, C, D and E are points on the circumference.  Calculate, giving reasons, the sizes of: a) $\hat{D}_1$ b) $\hat{O}_1$ c) $\hat{A}_1$ EC 2015	Ask: *Is there is any information in the question that should be transferred onto the diagram or anything you can fill in that you already know?* (Equal radii could be marked as such). Remind learners again that they could be required to use any theorem they have ever learned at any time – including theorems from Grade 8. Advice for learners: *Always fill in angles as you find them within the solution. The more angles that are filled in on the diagram, the easier the later questions will seem.*
Solution: a) $\hat{D}_1 = 25^o$ b) $\hat{O}_1 = 50^o$ c) $\hat{A}_1 = 25^o$	 <'s opp equal sides (radii) ext < of Δ <'s in same segment

15. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

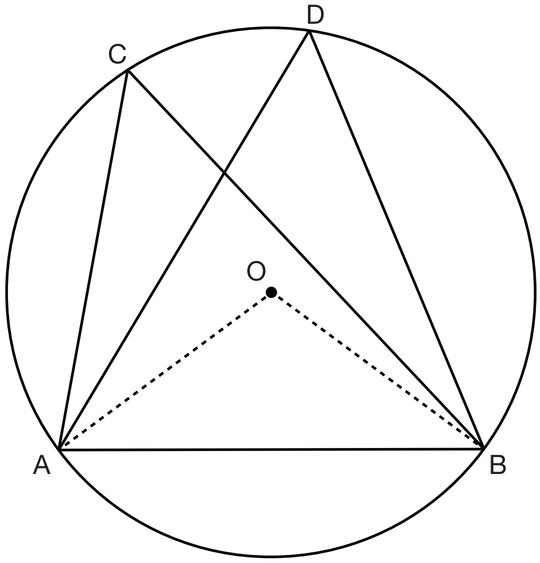
16. Give learners an exercise (from their textbooks or any other suitable exercise) to complete on their own.

17. Walk around the classroom as learners do the exercise. Support learners where necessary.

18. Tell learners that you are going to prove the theorem that they have just learned and practiced.

19. Ask learners to write the proof of the theorem in the back of their books so that all the proofs they need to learn for the exams are in one place. The heading should be the theorem written in full.

20. <u>Proof for: Angles subtended by a chord of the circle, on the same side of a chord, are equal.</u>



GIVEN: Circle with centre O and A, B, C and D all points on the circumference.

RTP: $\hat{C} = \hat{D}$

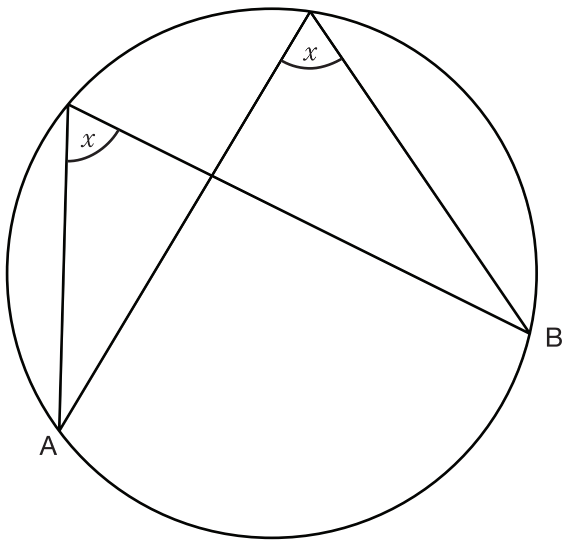
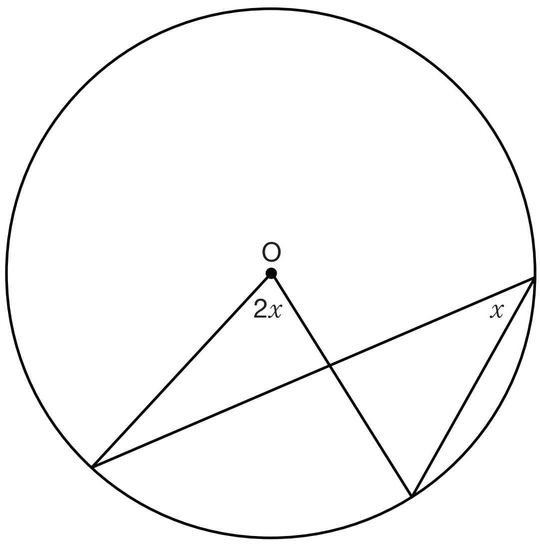
PROOF: Join OA and OB

Statement	Reason
$A\hat{O}B = 2\hat{C}$ $A\hat{O}B = 2\hat{D}$ $\therefore 2\hat{B} = 2\hat{D}$ $\therefore \hat{B} = \hat{D}$	< at centre $= 2 \times <$ at circumference < at centre $= 2 \times <$ at circumference

21. Ask learners if they have any questions.

22. Share with learners: Once this theorem has been learned, it is a common error to confuse it with the theorem regarding the angle at the centre. Learners begin to see the bowtie shape and assume angles in the same segment.

    Use the following diagrams as you discuss the issue:

	
Note that both angles are subtended at the circumference	Note that one angle is subtended at the centre and one angle is subtended at the circumference

23. Remind learners that they will need their drawn circles again for the next lesson. They should draw new ones if necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=jp8qjTscjZA

https://www.youtube.com/watch?v=_6QKO7NaYOQ

Term 3, Topic 2, Lesson 5

CYCLIC QUADRILATERALS

Suggested lesson duration: 2,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners should be able to: ● prove the 'opposite angle of a cyclic quad' theorem ● answer riders on the theorems, opposite angles of a cyclic quad and exterior angle of a cyclic quad ● prove that a quadrilateral is a cyclic quadrilateral.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 10 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw a quadrilateral and a circle.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
6 7	223 224	3 4	198 200	4 5 6	216 217 219	9.5	300	8.4	352

CONCEPTUAL DEVELOPMENT

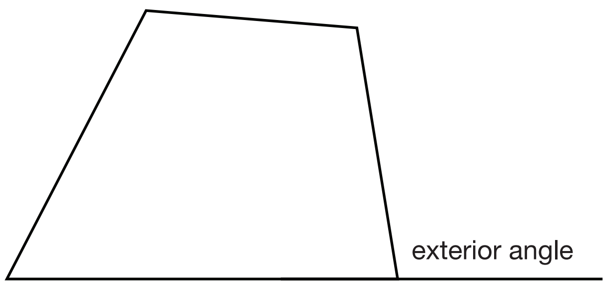
C

INTRODUCTION

1. Ensure learners have their two circles ready – these will be used for learners to measure and understand the theorem themselves.
2. While giving the instructions to learners to draw, measure or mark off on their own circle, walk around the class to assist where necessary.
3. Learners should work in pencil, so they can erase after the lesson then use the circles in all the lessons that follow while learning new theorems.

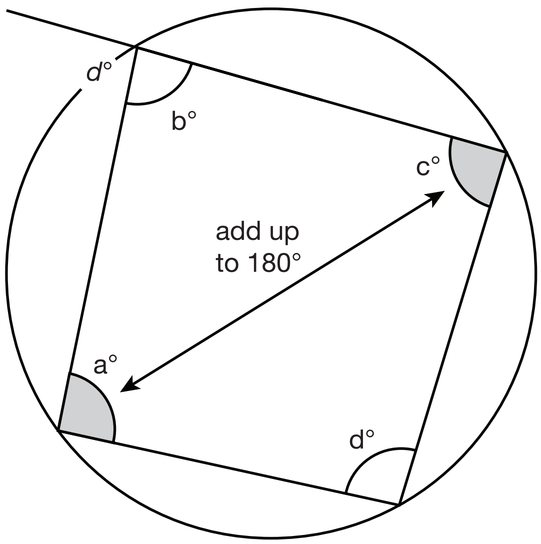
DIRECT INSTRUCTION

1. Start the lesson by asking: *What is a quadrilateral?* (A four-sided shape).
2. Ask learners to:
   - draw a quadrilateral in their exercise books
   - draw in one exterior angle of the quadrilateral.
3. Ensure learners know what the terms quadrilateral and exterior angle mean. Extend one of the sides on the quadrilateral drawn on the board to show an exterior angle. Ask learners to label the exterior angle they drew in their own books.



4. Ask: *What do think a cyclic quadrilateral might be?*
   (A quadrilateral where all four points lie on the circumference of a circle).
5. Tell learners: *Draw a cyclic quadrilateral in one of your drawn circles and measure all four angles*. Ask: *What do you notice?* (The opposite angles add up to 180°).
6. Tell learners to:
   a) extend one of the sides of their quadrilateral and measure the exterior angle
   b) extend a different side and measure another exterior angle.

   Ask: *What do you notice?*
   (The exterior angle is the same size as the opposite interior angle)
7. Learners should draw a sketch demonstrating both theorems in the same sketch.
   The sketch should look similar to this:



8. Ask learners to write the theorems in full into their books as well as the acceptable abbreviated form that can be used when using the theorem in a question. Tell them to copy the following table into their exercise books. Note that an extra two rows are added at the bottom of the table. These rows will be filled as the lesson proceeds.

Theorem	Acceptable abbreviated form
The opposite angles of a cyclic quadrilateral are supplementary.	opp <'s of cyclic quad
The exterior angle of a cyclic quad is equal to the opposite interior angle.	ext < of cyclic quad

9. Ask: *What do you think the converse statements to the two theorems could be?*
(If the opposite angles are supplementary, the quadrilateral must be cyclic;
If the exterior angle of a quadrilateral is equal to the opposite interior angle then the quadrilateral must be cyclic).

10. Tell learners to add these two theorems to the table now.

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.	converse opp <'s cyclic quad
If the exterior angle of a quadrilateral is equal to the opposite interior angle of the quadrilateral, then the quadrilateral is cyclic.	converse ext < of cyclic quad

11. Ask learners to highlight this summary in a way that it will be easy for them to find later when many theorems have been covered.

12. Before doing some examples, tell learners that a common question relating to these theorems is to prove that a quadrilateral is a cyclic quadrilateral. Ask learners to note how the two converse theorems they have just written down would assist them in proving that a quadrilateral is a cyclic quadrilateral.

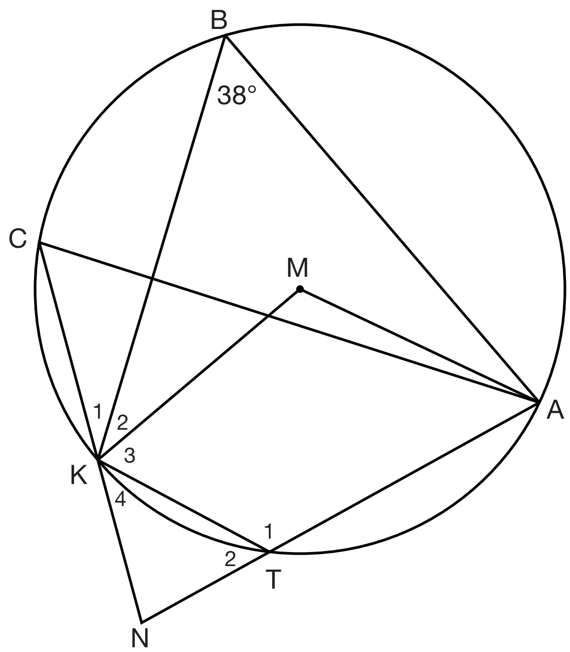
13. Tell learners that there is a third way of proving that a quadrilateral is a cyclic quadrilateral. Ask: *What can you remember from a past lesson that could prove a quadrilateral to be a cyclic quadrilateral*? Remind learners to turn back in their exercise books to look for the answer if necessary.
(If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic).

14. Summarise the three ways to prove that a quadrilateral is a cyclic quadrilateral on the board for learners to write in their exercise books.

How to prove that a quadrilateral is a cyclic quadrilateral: ● Exterior angle equals opposite interior angle ● One pair of opposite angles are supplementary ● Line segment subtends two equal angles.

15. Do two worked examples with learners. Learners should write the worked examples in their exercise books.

The diagrams for each of the worked examples are provided in the Resource Pack. Resource 10.

Example 1	Teaching notes
In the diagram, M is the centre of the circle. A, B, C, K and T lie on the circle. AT produced and CK produced meet in N. Also, NA = NC and $\hat{B} = 38°$ 	Ask: *Is there is any information in the question that needs to be transferred onto the diagram or anything you can fill in that you already know?* (NA = NC and equal radii (MK and MA) could be marked as such) Tell learners: *Start looking for angles you can calculate the sizes of using the theorems they have learned.* Remind learners: *If you fill the angles in on the diagram you must remember that the angles haven't been given in the information and therefore you need to give reasons how you knew the measure of the angles.* Tell learners: *You also need to look out for cyclic quads.* Ask: *Name any cyclic quads in this diagram.* (ABKT and ACKT.)
a) Calculate, with reasons, the size of the following angles: (i) $K\hat{M}A$ (ii) $\hat{T}_2$ (iii) $\hat{C}$ (iv) $\hat{K}_4$	It may be a good idea to shade these two cyclic quads – preferably in a slightly different way. Remind learners what you have told them before: *Always fill in angles as you find them within the solution.*

b) Show that NK = NT.	Say: *Ask yourself what needs to be true for two sides of a triangle to be equal in length?* (the opposite angles must be equal). This is usually the key to proving that two sides are equal. Note: sometimes congruency needs to be used.
c) Prove that AMKN is a cyclic quadrilateral. 2013 Exemplar	Ask learners to look back at the summary they made and remind you what the three possible options are. As each one is mentioned, ask learners if that one looks possible or not. Highlight the outer edge of the quadrilateral in question so it stands out more and have all known angles filled in.
Solutions: a) (i) $K\hat{M}A = 76^\circ$ (ii) $\hat{T}_s = 38^\circ$ (iii) $\hat{C} = 38^\circ$ (iv) $C\hat{A}N = 38^\circ$ $\therefore \hat{K}_4 = 38^\circ$ b) $\hat{K}_4 = \hat{T}_2$ $\therefore$ NK = NT c) $\hat{N} = 104^\circ$ $\therefore$ AMKN is a cyclic quad	< at centre $= 2 \times <$ at circumference ext < cyclic quad <'s in same segment Point out to learners that there was a second option here – ext < of cyclic quad – remind them that they can use any one. <'s opp equal sides (NA = NC) ext < cyclic quad proved above sides opp equal <'s <'s of Δ converse opp <'s cyclic quad

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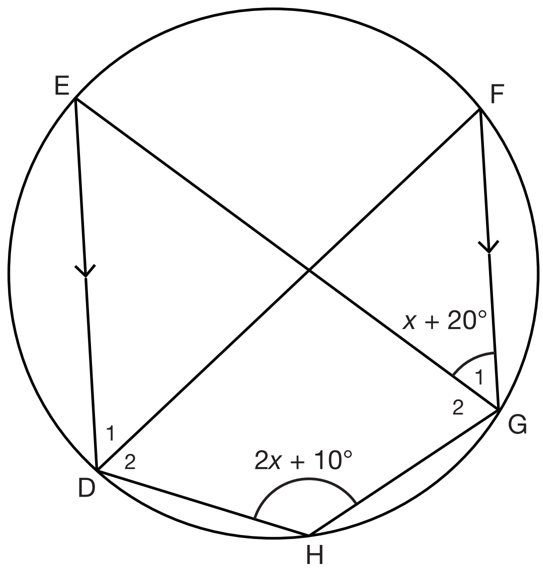
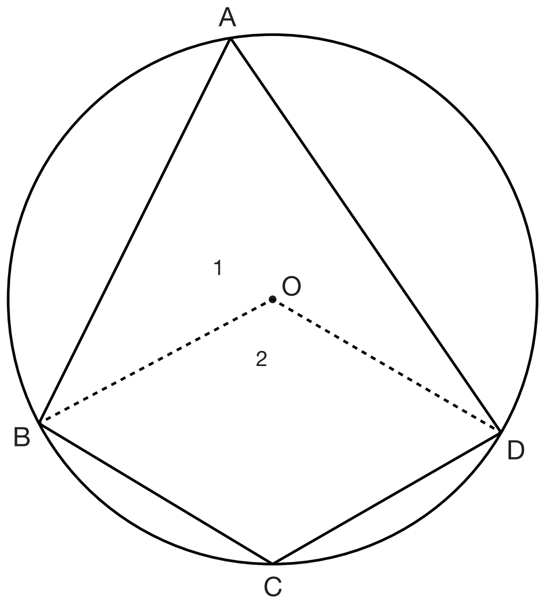
Example 2	**Teaching notes**
D, E, F, G and H are points on the circumference of a circle. $\hat{G}_1 = x + 20^o$ and $\hat{H} = 2x + 10^o$. $DE//FG$.  a) Determine the size of $D\hat{E}G$ in terms of x. b) Calculate the size of $D\hat{H}G$ NSC NOV 2016	Ask: *Is there is any information in the question that needs to be transferred onto the diagram or anything you can fill in that you already know?* Point out the parallel lines mentioned. Remind learners that information is not given unless it is useful. Learners will therefore be expected to use at least one of the parallel line theorems that they learned in Grade 8. Ask: *What parallel line theorems did you learn about in Grade 8?* *Ask: Can you see any cyclic quads? If yes, name them.* (DFGH and EDHG).
Solution: a) $D\hat{E}G = x + 20^o$ b) $2x + 10^o + x + 20^o = 180^o$ $3x + 30^o = 180^o$ $3x = 150^o$ $x = 50^o$ $\therefore D\hat{H}G = 2(50^o) + 10^o$ $\therefore D\hat{H}G = 110^o$	 $DE//FG$; alt <’s equal opp <’s cyclic quad

16. Ask directed questions so that you can ascertain learners’ level of understanding.
    Ask learners if they have any questions.
17. Give learners an exercise (from their textbook or any other relevant exercise) to complete on their own.
18. Walk around the classroom as learners do the exercise. Support learners where necessary.
19. Prove one of the theorems that they have just learned and practiced.

20. Ask learners to write the proof of the theorem at the back of their exercise books so that all the proofs they need to learn for the exams are in one place. The heading should be the theorem written in full.

21. Proof for: The opposite angles of a cyclic quadrilateral are supplementary.



GIVEN: A, B, C and D are points that lie on the circumference of a circle

RTP: $\hat{A} + \hat{C} = 180^o$ and $\hat{B} + \hat{D} = 180^o$

PROOF: Join OB and OD. Let $B\hat{O}D = \hat{O}_2$ and reflex $B\hat{O}D = \hat{O}_1$

Statement	**Reason**
$\hat{O}_2 = 2\hat{A}$	< at centre at circumference
$\hat{O}_1 = 2\hat{C}$	< at centre at circumference
$\therefore \hat{O}_2 + \hat{O}_1 = 2\hat{A} + 2\hat{C}$	
but $\hat{O}_2 + \hat{O}_1 = 360^o$	
$\therefore 2\hat{A} + 2\hat{C} = 360^o$	
$\therefore 2(\hat{A} + \hat{C}) = 360^o$	
$\therefore \hat{A} + \hat{C} = 180^o$	
Similarly, $\hat{B} + \hat{D} = 180^o$ by joining AO and OC.	

22. Ask learners if they have any questions.

23. Remind learners that they will need their drawn circles again for the next lesson.

D

ADDITIONAL ACTIVITIES/READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=xMrAVn7am5M
(Understand the meaning of a cyclic quad)

https://www.youtube.com/watch?v=aNLwD4yyL0I
(Proof of opposite angles of a cyclic quad)

Term 3, Topic 2, Lesson 6

TANGENTS

Suggested lesson duration: 2,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	34-36

Lesson Objectives

By the end of the lesson, learners should be able to:

- prove the 'alternate segment' (tan-chord) theorem and the 'equal tangents' theorem
- answer riders on the theorems, equal tangents from a common point, tangent perpendicular to radius and angle between a tangent to a circle and a chord
- prove that a line is a tangent to a circle.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resources 11 and 12 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw three circles.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
8 9 10	226 227 229	5	205	7 8	220 222	9.6	308	8.5 8.6	356 359

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ensure learners have their two circles ready – these will be used for learners to measure and understand the theorem themselves.
2. While giving the instructions to learners to draw, measure or mark off on their own circle, walk around the class to assist where necessary.

DIRECT INSTRUCTION

1. Ask learners to:
   a) Mark the centre of the circle and draw in the radius on one of your circles.
   b) Draw a tangent to the circle. The point of tangency must be where the radius meets the circumference.
   c) Repeat the above steps in a different part of the circle so that you have two tangents and two radii drawn in.
   d) Measure the angle made by the radius and tangent.
   Ask: *What do you notice about the angle made by the radius and tangent?* (it is 90°).
   Tell learners to look at another learner's diagram to confirm whether they got the same result.
2. Ask learners to:
   a) Draw a chord in their second circle.
   b) Draw a tangent to the circle. The point of tangency must be where the chord meets the circumference.
   c) Subtend an angle from the chord into the opposite segment to where the tangent lies (tell learners they should have a triangle and a tangent now).
   d) Measure the angle between the tangent and the chord. Fill in the measurement.
   e) Measure the angle subtended from the chord.
   Ask: *What do you notice about the angle subtended from the chord?*
   (The angles are equal). Tell learners to look at another learner's diagram and confirm whether they got the same result.

3. Ask learners to go back to their first circle where they had a tangents and radii drawn in and adapt it as follows:

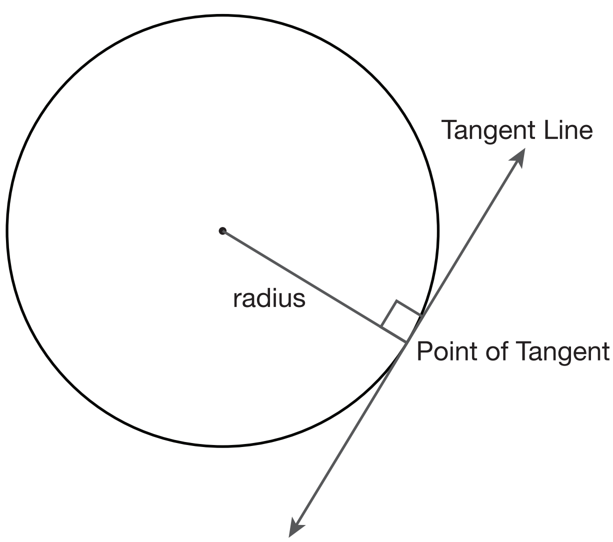
   a) Lengthen the tangents (in the direction that makes most sense) until they intersect.

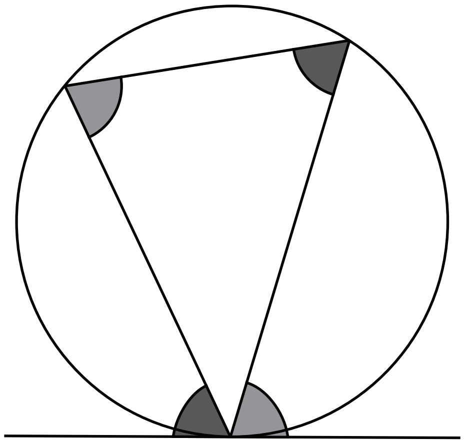
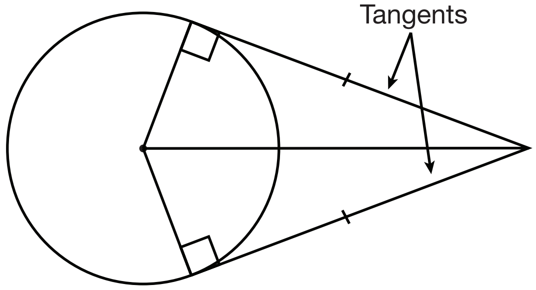
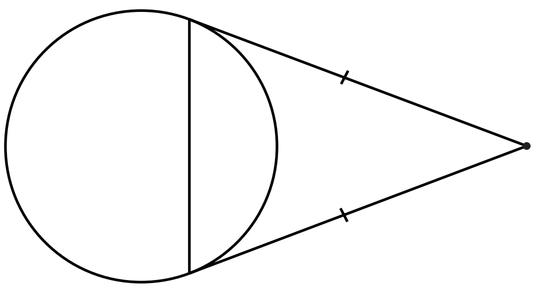
   b) Measure the length of each tangent from the point of tangency to the point of intersection.
   Ask: *What do you notice about the length of each tangent from the point of tangency to the point of intersection?* (They are equal in length). Tell learners to look at another learner's diagram and confirm whether they got the same result.

4. Tell learners that these are the three theorems that we will be learning about today. They are the last three circle theorems.

5. Use the three circles you have on the board. Summarise the three theorems. Ask learners to draw the diagrams in their exercise books. The summary should look as follows. Use the teaching notes to discuss the theorems as you draw each sketch.

The following diagrams are available in the Resource Pack. Resource 11.

	Teaching notes
	Point out to learners that no matter where the tangent is, it will always be perpendicular to a radius that it meets at the circumference.

	Learners find this theorem challenging. Show learners to first look what angle is made by the tangent and the chord by moving their finger from the tangent to the chord. Once the chord is established, put a finger at each end of the chord and see which angle it subtends. Encourage learners to do the same. Perhaps ask for a few volunteers to come and demonstrate.
 	Learners must recognise that both diagrams alongside show the same theorem. During the investigation learners produced the first diagram where the tangents were connected to the radii. It is important that they don't think it always needs to look like this to still be true. The second diagram, showing the tangents connected to a chord, is important as it produces an isosceles triangle. This appears often in questions.

6. Ask learners to write the theorems in full into their exercise books as well as the acceptable abbreviated form that can be used when using the theorem in a question. Tell learners to copy the following table into their books. Note that an extra two rows have been added at the bottom of the table. These rows will be filled as the lesson proceeds.

Theorem	**Acceptable abbreviated form**
A tangent to a circle is perpendicular to the radius at the point of contact.	rad⊥tan
Two tangents drawn to a circle from the same point outside the circle are equal in length	tans from common point
The angle between the tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem

7. Tell learners that only the first and last of these three theorems have a converse.
   Ask: *What do you think the converse statements to first of the two theorems could be?*
   (If a line is perpendicular to the radius at the point where it meets the circumference, then the line is a tangent).

8. The converse of the tan-chord theorem is more complicated and may be difficult for learners to verbalise. If the angle formed by a chord and a line drawn at its endpoint is equal to an angle subtended by the chord in the opposite segment, then the line must be a tangent to the circle.

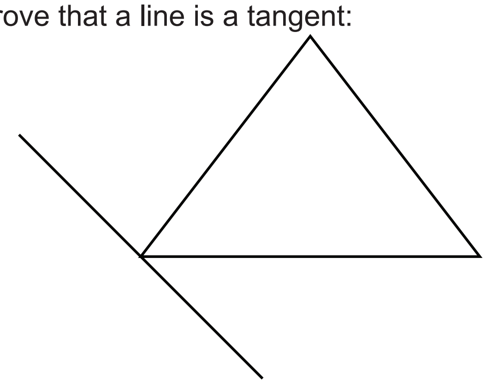
9. Tell learners to add these two theorems to the table now.

If a line is drawn perpendicular to the radius at the point where the radius meets the circle, then the line is a tangent to the circle	converse tan⊥rad
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem

10. Ask learners to highlight this summary in a way that makes it easy for them to find later when many theorems have been covered.

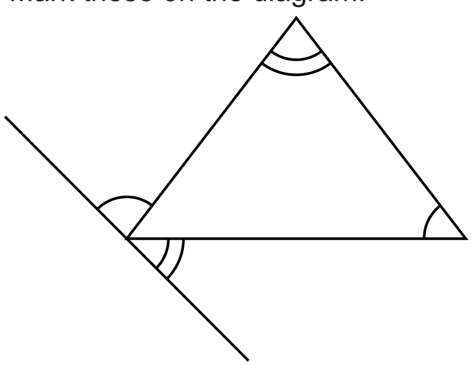
11. Before doing two worked examples with learners, go back to the converse of the tan-chord theorem and discuss it further. This theorem is commonly asked in many assessments. It involves proving that a line is a tangent to three points (forming a triangle).

12. Ask learners to write the heading: ‘How to prove a line is a tangent to 3 points’ in their books and to copy the following diagram.
    This is what learners will see when they are asked to prove that a line is a tangent:



Ask a volunteer to come to the board and show you which angles should be equal IF the line IS a tangent.

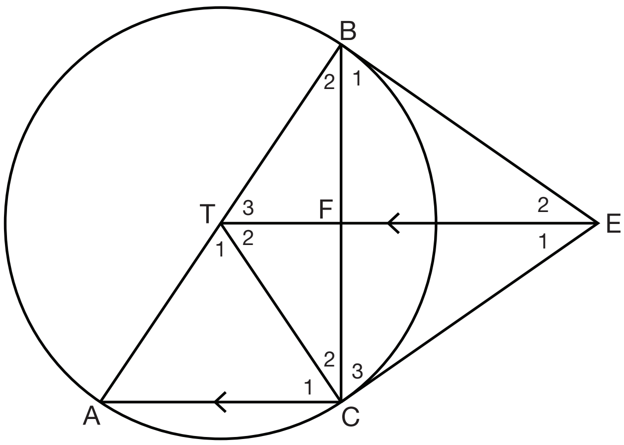
Mark these on the diagram:



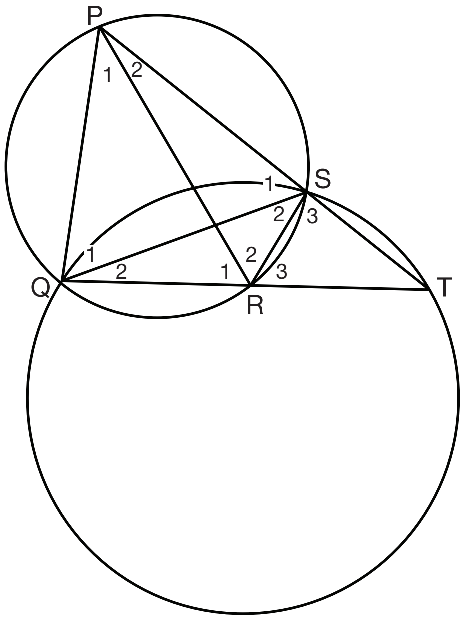
Tell learners that this is what they should do in an assessment – shade the triangle mentioned then decide which angles should be equal. This will focus learners on what needs to be proved. Remind learners that they need only find one pair of equal angles.

13. Do two worked examples with learners now. Learners should write the worked examples in their exercise books.

The diagrams for each of the worked examples are provided in the Resource Pack. Resource 12.

Example 1	Teaching notes
In the diagram, the vertices A, B and C of ABC are concyclic. EB and EC are tangents to the circle at B and C respectively. T is a point on AB such that $TE//AC$. BC cuts TE in F.  a) Prove that $\hat{B}_1 = \hat{T}_3$	Ask: *Is there is any information in the question that needs to be transferred onto the diagram or anything you can fill in that you already know?* (Alternate angles equal; corresponding angles equal; tan-chord; equal tangents) Tell learners to be aware of what has been given and what has not. Many learners make the error of thinking $\hat{C} = 90^o$ because it 'looks like' AB is a diameter. Learners should already have marked $\hat{B}_1 = \hat{A}$ (tan chord) which leads to the corresponding angles on the parallel lines.
b) Prove that TBEC is a cyclic quadrilateral	Tell learners to shade the quad. Ask: *What are the three ways of proving a cyclic quad?* As each one is mentioned, discuss which method seems the most likely to be used.
c) Prove that ET bisects $B\hat{T}C$	Ask: *What does the term bisect mean?* Ask: *What angles do we need to prove equal to answer this question?* ($\hat{T}_2 = \hat{T}_3$). Remind learners that they have proved that TBEC is a cyclic quad and that this may be useful.

d) If it is given that TB is a tangent to the circle through B, F and E, prove that TB = TC..	Ask: *What angles need to be equal for TB to equal TC?* ($\hat{B}_2$ & $\hat{C}_2$). Remind learners again about the cyclic quad that has been proved.
e) Hence, prove that T is the centre of the circle through A, B and C. NSC NOV 2015	Discuss the word 'hence' with learners. Remind them that it means 'use what you have just proved'. In other words, the fact that BT=TC should be useful to answer this question.
Solution: a) $\hat{B}_1 = \hat{A}$ $\hat{T}_3 = \hat{A}$ $\therefore \hat{B}_1 = \hat{T}_3$ b) $BE = CE$ $\therefore \hat{B}_1 = \hat{C}_3$ and $\hat{B}_1 = \hat{T}_3$ $\therefore \hat{T}_3 = \hat{C}_3$ $\therefore$ TBEC is a cyclic quad c) $\hat{B}_1 = \hat{T}_3$ $\hat{B}_1 = \hat{T}_2$ $\therefore \hat{T}_2 = \hat{T}_3$ $\therefore$ ET bisects $B\hat{T}C$ d) $\hat{B}_2 = \hat{E}_2$ $\hat{C}_2 = \hat{E}_2$ $\therefore \hat{B}_2 = \hat{C}_2$ $\therefore TB = TC$ e) In ΔTAC: $\hat{A} = \hat{T}_3$ $\hat{T}_3 = \hat{T}_2$ $\hat{T}_2 = \hat{C}_1$ $\therefore \hat{A} = \hat{C}_1$ $\therefore TA = TC$ But $TC = TB$ $\therefore TA = TC = TB$ $\therefore$T is the centre of the circle	 tan-chord theorem $TE // AC$; corres <'s tans from common pt <'s opp equal sides converse <'s same segment proved above <'s in same segment tan chord theorem <'s in same segment sides opp equal <'s $TE // AC$; corres <'s Proved above $TE // AC$; alt <'s sides opp equal <'s

Example 2	**Teaching notes**
In the diagram, PQ is a tangent to the circle QST at Q such that QT is a chord of the circle and TS produced meets the tangent at P. R is a point on QT such that PQRS is a cyclic quadrilateral in another circle. PR, QS and RS are joined. 	Ask: *Is there is any information in the question that needs to be transferred onto the diagram or anything you can fill in that you already know?* (tan-chord; angles in same segment)
a) Give a reason for each statement below: (i) $\hat{Q}_1 = \hat{T}$ (ii) $\hat{Q}_2 = \hat{P}_2$	Learners should find this relatively easy
b) Prove that PQR is an isosceles triangle.	Ask: *What angles need to be equal to make this triangle isosceles?* ($P\hat{Q}R = \hat{R}_1$) Tell learners that as none of the angles have been given variables it would be a good idea to use this method. Let the angles found equal in a) be given a variable each. This should help learners to notice that the angle at Q is made up of both variables. Learners now need to find a theorem that will help them find that the angle at R is also equal to the sum of these variables.

c) Prove that PR is a tangent to the circle RST at point R. NSC NOV 2014	Advise learners to shade the triangle given and highlight the line. Remind learners what was discussed in class earlier and tell them to look back in their books to confirm that the diagram in this question looks like one of the diagrams they drew in their books. Point out that the line doesn't cross over both sides in this case which makes it easier – there is only one possible pair of angles for them to focus on. Ask: *Which pair of angles should you focus on?* ($\hat{R}_2 = \hat{T}$)
Solution: a) (i) tan-chord theorem (ii) <'s in same segment b) Let $\hat{Q}_1 = \hat{T} = a$ and $\hat{Q}_2 = \hat{P}_2 = b$ $\therefore P\hat{Q}R = a + b$ $\hat{R}_1 = a + b$ $\therefore P\hat{Q}R = \hat{R}_1$ $\therefore PQ = PR$ $\therefore \Delta PQR$ is isosceles c) $\hat{R}_2 = \hat{Q}_1 = a$ $\hat{T} = a$ $\therefore \hat{R}_2 = \hat{T}$ $\therefore$ PR is a tangent to the circle RST at point R	 ext < of ΔPRT

14. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

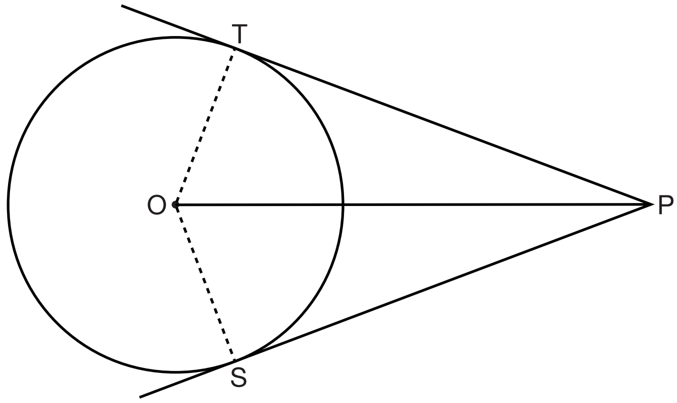
15. Give learners an exercise (from their textbook or any other relevant exercise) to complete on their own.

16. Walk around the classroom as learners do the exercise. Support learners where necessary.

17. Tell learners that now you are going to prove two of the theorems that they have just learned and practiced.

18. Ask learners to write the proofs of the theorems at the back of their exercise books so that all the proofs they need to learn for the exams are in one place. The heading should be the theorem written in full.

19. Proof for: Two tangents drawn to a circle from the same point outside the circle are equal in length.



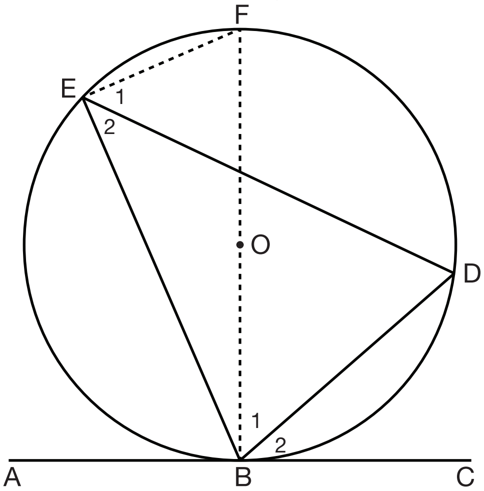
GIVEN: Circle with two tangents, PT and PS.

RTP: PT = PS

PROOF: Join OT and OS

Statement	Reason
In ΔOTP and ΔOSP	
$OT = OS$	radii
$OP = OP$	common
$O\hat{T}P = O\hat{S}P = 90^o$	tan⊥rad
$\therefore \Delta OTP \equiv \Delta OSP$	RHS
$\therefore PT = PS$	

20. Proof for: The angle between the tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment.



GIVEN: Circle with centre O and tangent ABC

RTP: $C\hat{B}D = B\hat{E}D$

PROOF: Draw diameter BOF and join EF.

Statement	Reason
$\hat{B}_1 + \hat{B}_2 = 90^o$	tan⊥rad
$\hat{E}_1 + \hat{E}_2 = 90^o$	< in semi-circle
$\therefore \hat{B}_1 + \hat{B}_2 = \hat{E}_1 + \hat{E}_2$	
but $\hat{B}_1 = \hat{E}_1$	<'s in same segment
$\therefore \hat{B}_2 = \hat{E}_2$	
$\therefore C\hat{B}D = B\hat{E}D$	

21. Ask learners if they have any questions.

22. Tell learners that all the new theorems have now been covered. Remind them that they should have 21 theorems in total (including the converses and corollaries). The proofs of seven theorems are required for examination purposes. Some proofs are asked more often than others. List those asked most often for learners with one or two words that can be used as a hint as to how to go about the proof. Tell learners they need to know their proofs well and if they remember the hint words, it would help them on the day of the examination. Tell learners to write this in their exercise books.

Theorem	Hint for proof
The line drawn from the centre perpendicular to the chord bisects the chord	Congruency (draw in radii)
Angle at centre equals 2 times angle at circumference	Ext angle of Δ (draw in diameter)
Opposite angles of a cyclic quad are supplementary	Angle at centre and angles around a point (draw in radii)
The angle between the tangent and chord is equal to the angle in the opposite segment	90° (semi-circle and rad/tan) (draw in diameter and new chord to form 'bowtie')

23. Tell learners that in the next lesson they will start practicing questions that will bring all their newly gained knowledge together.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=U33XHR9faUE
(Proof of tan chord theorem)

Term 3, Topic 2, Lesson 7

COMBINATION OF ALL THEOREMS AND CONSOLIDATION

Suggested lesson duration: 3 hours

A POLICY AND OUTCOMES

CAPS Page Number	34-36
Lesson Objectives By the end of the lesson, learners should be able to: • answer riders covering all the circle theorems.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resources 13, 14 and 15 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw the first diagrams for the first example.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
11 12 13 Rev Some Ch	236 240 242 243 245	6 7 Rev	206 209 211	Qu's	224	9.7 9.8 Rev	319 323 325	8.7	366

CONCEPTUAL DEVELOPMENT

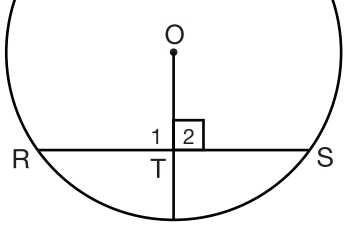
C

INTRODUCTION

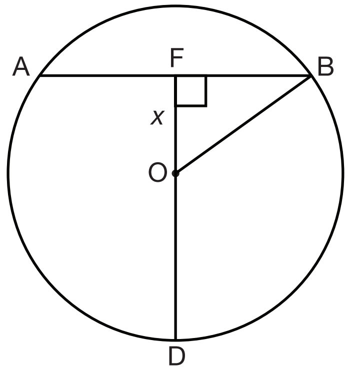
1. This lesson is important. Thus far learners have mostly used the theorems in isolation. They must now combine all the information and consolidate their knowledge.
2. Encourage learners. Enjoying the challenge of Euclidean Geometry will go a long way to improving their ability in it.

DIRECT INSTRUCTION

1. Ask learners to recap what they have learned in this section. Spend time pointing out issues that you know are important as well as problems that you encountered from your own learners during the topic.
2. If learners want you to explain a concept again, do that now.
3. Tell learners that you are going to do all the geometry questions from the final examination from 2017 before they do a consolidation exercise.
4. As you do each of the examples, get as much input from learners as possible.
   Ask questions and encourage discussion. Stop regularly and ask if there are any questions.
   Learners must write each worked example in their exercise books.
5. All the diagrams are available and enlarged in the Resource Pack. Resources 13-15.

Example 1	**Teaching notes:**
a) In the diagram below, O is the centre of the circle and point T lies on chord RS. Prove the theorem which states that if $OT \perp RTS$ then $RT = TS$ 	Tell learners if they know their theory this question should present no problems. Learners should remember that even if they don't manage to prove a theorem asked, they should still attempt the second part of the question which will be directly related to the theorem that needs to be proved.

b) In the diagram, O is the centre of circle ABD. F is a point on chord AB such that $DOF \perp AB$. $AB = FD = 8cm$ and $OF = x$ cm. Determine the length of the radius of the circle.



Advise learners to read the question carefully and fill in any possible information onto the diagram.

Ask: *What do you know about AB if a line is drawn from the centre that is perpendicular to the chord?* ($AF = FB$).

As the length of AB was given, this means AF and FB are also known.

Ask if that helps? If learners are struggling, ask: *Do know the length of OD which is the radius and is therefore equal to OB?*

Learners may not know OD directly but it can be given as an expression because the length of FD was given and OF is 'known'.

If an equation can be made with only one unknown then it can be solved.

Solution:

a) GIVEN: Circle with centre O and chord RS. $OT \perp RS$.

RTP: $RT = TS$

PROOF:

Statement	Reason
Join OR and OS	
In ΔROT and ΔSOT	
$OR = OS$	Radii
$OT = OT$	Common
$\hat{T}_1 = \hat{T}_2 = 90^o$	Given
$\therefore \Delta ROT \equiv \Delta SOT$	RHS
$\therefore RT = TS$	

b) $AF = FB = 4$ cm — line from centre $\perp$ to chord

$OD = 8 - x$

$\therefore OB = 8 - x$ — radii

$OB^2 = OF^2 + BF^2$ — Pythagoras

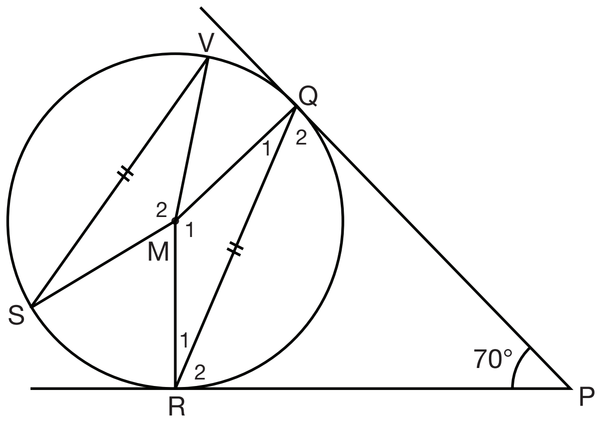
$(8-x)^2 = x^2 + 4^2$

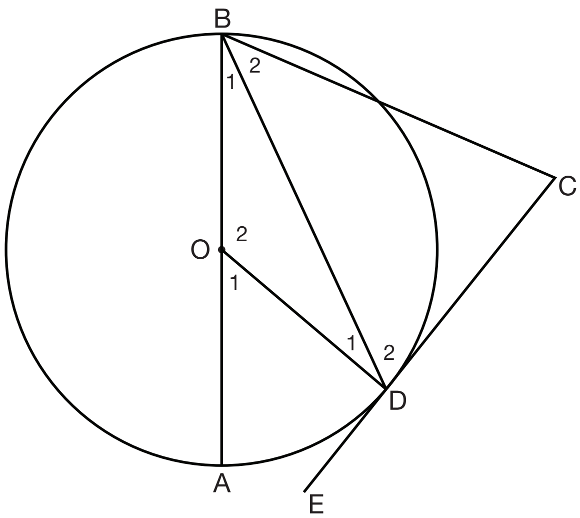
$64 - 16x + x^2 = x^2 + 16$

$-16x = -48$

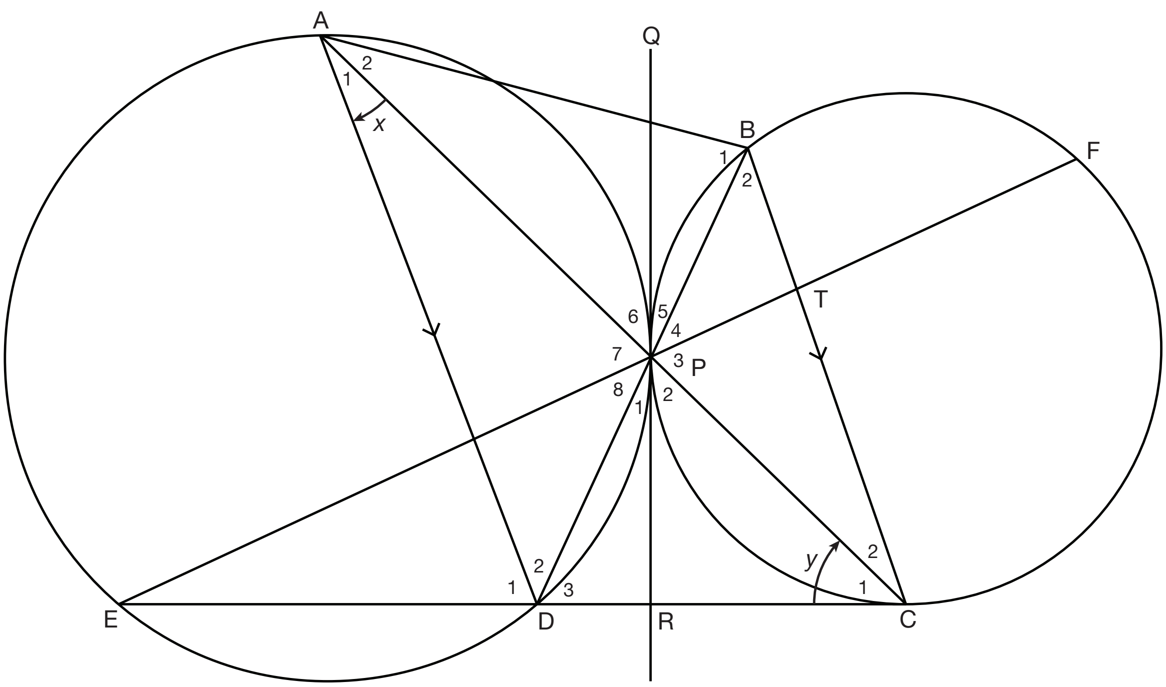
$x = 3$

$\therefore r = 8 - 3 = 5$ cm

Example 2	**Teaching notes:**
M is the centre of the circle SVQR having equal chords SV and QR. RP and QP are tangents to the circle at R and Q respectively such that $R\hat{P}Q = 70^o$.  a) Calculate the size of $\hat{R}_2$.	Remind learners to read the question carefully and to fill in any possible information onto the diagram. (Centre given therefore equal radii; equal tangents from common point; $\hat{M}_1 = \hat{M}_2$ because of equal angles from equal chords; a radius meeting a tangent therefore there are right angles). Note that the equal tangents form an isosceles triangle.
b) Calculate the size of $\hat{Q}_1$.	$\hat{Q}_2$ will be known if $\hat{R}_2$ is known and $MQ \perp QP$
c) Determine the size of $\hat{M}_2$.	If $\hat{Q}_1$ is known, $\hat{R}_1$ is known (isosceles) and therefore $\hat{M}_1$ can be calculated (<'s of triangle). This leads directly to $\hat{M}_2$.
Solution: a) $PQ = PR$ $\therefore \hat{R}_2 = \hat{Q}_2$ $\hat{R}_2 + \hat{Q}_2 + 70^o = 180^o$ $\therefore \hat{R}_2 = 55^o$ b) $\hat{Q}_2 = 55^o$ $\hat{Q}_2 + \hat{Q}_1 = 90^o$ $\therefore \hat{Q}_1 = 35^o$ c) $\hat{R}_1 = 35^o$ $\hat{M}_1 + 35^o + 35^o = 180^o$ $\therefore \hat{M}_1 = 110°$ $\therefore \hat{M}_2 = 110°$	 tans from common pt <'s opp equal sides <'s of Δ proved above tan⊥rad <'s opp equal sides <'s of Δ equal chords; equal <'s

Example 3	**Teaching notes:**
In the diagram below, O is the centre of the circle. CDE is a tangent to the circle at D. DB bisects $A\hat{B}C$. Let $\hat{B}_1 = x$.  	Remind learners to read the question carefully and to fill in any possible information onto the diagram. Ask: *What do you know because a tangent is given?* Ask: *What does the term bisect mean?* Ask: *What do you know if the centre has been given?* (Label $\hat{B}_1$; mark radius $\perp$ tangent; mark equal radii; mark $\hat{B}_2 = \hat{B}_1(x)$)
a) Prove that $BC//OD$.	Ask: *What would be true if the lines were parallel?* This should give a clue of what angles need to be proved equal. Equal angles: $\hat{B}_2$ and $\hat{D}_1$ OR $\hat{O}_1$ and $\hat{B}_2 + \hat{B}_1$ Supplementary angles: $\hat{B}_2; \hat{B}_1; \hat{O}_2$ OR $\hat{D}_1; \hat{D}_2; \hat{C}$ As each of these are given, discuss the merits and possibility of each one.
b) Show that $\hat{C} = 90^o$.	Remind learners that by now the lines will have been proved parallel. Ask how this helps. (tan$\perp$rad; co-int <’s)
Solution: a) $\hat{D}_1 = x$ $\hat{B}_2 = x$ $\therefore \hat{B}_2 = \hat{D}_1$ $\therefore BC//OD$ b) $O\hat{D}C = 90^o$ $\therefore \hat{C} = 90^o$	 <’s opp equal sides Given alt <’s equal tan$\perp$rad $BC//OD$; co-int <’s

Example 4	Teaching notes:
In the diagram below two circles touch each other externally at point P. QPR is a common tangent to both circles at P. EDRC is a tangent to circle PBFC at C. $\text{R}\hat{\text{C}}\text{A} = y$ and $\text{D}\hat{\text{A}}\text{C} = x$. $\text{AD}//\text{BC}$. a) Name, with reasons, 4 other angles equal to x. b) Show that $\text{E}\hat{\text{P}}\text{A} = x + y$. c) Determine the numerical value of $x + y$ if it is given that DCTP is a cyclic quadrilateral.	Point out to learners that this is the last question in the exam for Euclidean Geometry and will always be challenging. Learners should remember that they have the knowledge and they know the theorems to assist them. Even if the diagram looks complicated (as this one does), they need to read through each piece of information carefully and focus on one piece of information at a time, marking it on their diagram and working through the information, asking themselves what else they know. It would be useful to use some colour to mark the diagram.



Learners need to be able to deconstruct a complex diagram in order to identify theorems that can be used.

The first piece of information is the common tangent. Learners should think about the three tangent theorems they know and mark any equal angles. As there is no centre, and therefore no radius marked, it is reasonable that they only need to focus on the tan-chord theorem. Due to the fact that it is a common tangent, focus on one circle at a time.

The second piece of information is also a tangent. Learners should focus only on that and think through the three theorems and mark any equal angles.

The third piece of information is about parallel lines. Learners should think about the three theorems they know dealing with parallel lines and mark any information as such.

Use the x and y labels where possible.

Remind learners to remember their other theorems also. (Any circle theorems that don't require the centre may be useful as well as their Grade 8 theorems).

a) If learners have marked all equal angles that they know they should have found all four.

b) This may be a little difficult for some learners to see. It is useful to remember Grade 8 theorems and to ask themselves if there is any connection. If any learners notice that $\hat{P}_7$ is in fact the exterior angle to ΔPEC, praise them.

c) Learners should focus their attention on what they know about cyclic quadrilaterals – both theorems will be useful. Advise learners to shade the cyclic quadrilateral so it stands out. This may also help them to notice exterior angles. Since a numerical value is required (and none were given), this implies that an equation will be needed in order to solve for the variables $x+y$.

Solution:	
a) $\hat{C}_2 = x$	$AD//BC$; alt <'s equal
$\hat{P}_1 = x$	tan chord theorem
$\hat{P}_5 = x$	vert opp <'s equal
$\hat{E} = x$	<'s in same segment
	Point out that the reasons could have been different depending on the order of angles used. For example, after stating that $\hat{C}_2 = x$, $\hat{P}_5$ could have been found using the tan-chord theorem
b) $\hat{P}_7 = \hat{E} + \hat{C}_1$	ext < of Δ
$\therefore \hat{P}_7 = x + y$	
c) $\hat{P}_8 = x + y$	ext < cyclic quad
$\hat{P}_1 + \hat{P}_2 + \hat{P}_8 + \hat{P}_7 = 180°$	<'s on str line
$\therefore x + y + x + y + x + y = 180°$	
$\therefore 3x + 3y = 180°$	
$\therefore 3(x + y) = 180°$	
$\therefore x + y = 60°$	
$\therefore x + y = 60^o$	

6. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
7. Give learners an exercise to complete with a partner before giving them another one to complete on their own. Remind learners that Euclidean geometry requires continuous and deliberate practice – it cannot be mastered in a short space of time.
8. Walk around the classroom as learners do the exercise. Support learners where necessary.
9. Once learners have completed the exercises provided and you have assisted them with any corrections, give the learners the following activity to do to assist them for the final examinations:

Tell learners to summarise all their theorems in one table (so far, they have multiple small tables). They need to add an extra column at the end where they should make a sketch to demonstrate the theorem.
Suggest that they put their Grade 8 and 9 theorems in the summary also.

Give learners a few days to a week to complete their summaries. Check that they have an accurate summary and that they have remembered all the theorems. Time permitting, learners could work in pairs or groups of three to check each other's summaries and add to their own if necessary. This should be a useful and productive experience that will assist learners in their understanding and preparation for an assessment.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=5WB1911rBz4
(A 25 minutes summary of all the theorems)

https://www.youtube.com/watch?v=r95396R1jj4
(A 30-minute video with many examples covered)

https://www.youtube.com/watch?v=XUus6-9E9sQ&t=59s
(Circle theorems in 3 minutes)

http://www.vocfm.co.za/wp-content/uploads/2014/10/GEOMETRY_GRADE-11_12.pdf
(A summary of the theorems with questions).

TOPIC OVERVIEW

TOPIC 3: TRIGONOMETRY

A TOPIC OVERVIEW

- This topic is the third of five topics in Term 3.
- This topic runs for two weeks (9 hours).
- It is presented over seven lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 9 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Trigonometry counts 33% of the final Paper 2 examination.
- All three proofs are required for examination purposes. In this set of lesson plans, the three rules are learned and practiced. Proofs are in a lesson of their own because learners need the time to comprehend what the rule is about before they work through the proof.

Breakdown of topic into 7 lessons:

	Lesson title	**Suggested time (hours)**		**Lesson title**	**Suggested time (hours)**
1	Revision	1	5	Proofs of the three rules and the ambiguous case of the sine rule	1
2	The Sine rule	1,5	6	2 dimensional questions involving all 3 rules	2
3	The Cosine rule	1,5	7	Revision and Consolidation	1
4	The Area rule	1			

SEQUENTIAL TABLE

B

GRADE 10 and Senior phase	GRADE 11	GRADE 12
LOOKING BACK	CURRENT	LOOKING FORWARD
• Solving of right-angled triangles using Gr 10 Trigonometry • Finding the area of right-angled triangles (Area $= \frac{1}{2}\text{base} \times \perp \text{height}$) • Basic Geometry theorems from previous years, particularly Grade 8 • Calculator work using Trigonometry functions • Knowledge/ understanding of angles of elevation and depression.	• Prove and apply the sine, cosine and area rules • Solve problems in two dimensions using the sine, cosine and area rules.	• Solve problems in 3 dimensions • Problems can include compound or double angles.

WHAT THE NSC DIAGNOSTIC REPORTS TELL US

C

According to **NSC Diagnostic Reports** there are several issues pertaining to solving triangles using Trigonometry.

These include:

- recognising right-angled triangles and applying the Theorem of Pythagoras
- difficulty in selecting the sides/angles required. Many fail to see that two triangles share a side or an angle
- difficulty in seeing which rule was required
- poor algebraic manipulation skills (changing the subject of the formula).

It is important that you, as the teacher, keep these issues in mind when teaching this section.

While teaching the 2D section of Trigonometry, it is important to keep in mind that 3D Trigonometry will be introduced in Grade 12. If learners have a good understanding of the three rules required, they will have a better chance of succeeding at this (perceived) difficult topic.

D

ASSESSMENT OF THE TOPIC

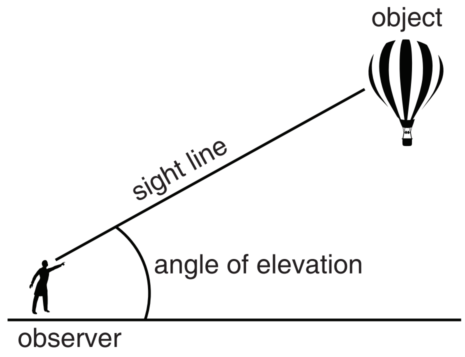
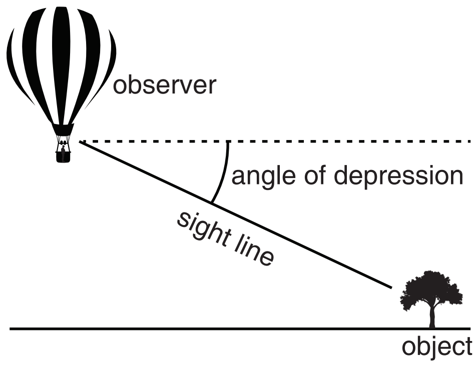
- CAPS formal assessment requirements for Term 3:
  - Two tests
- Two tests, with memorandum, are provided in the Resource pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of solving triangles, sometimes related to real-life situations. Proofs could also be asked.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

E

MATHEMATICAL VOCABULARY

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	**Explanation**
opposite	The side, in a triangle, opposite the angle of interest/the angle named
adjacent	The side, in a triangle, next to the angle of interest/the angle named
hypotenuse	The side, in right-angled triangle, opposite the right angle (always the longest side)
included angle	The angle between two sides. The angle formed by two sides
horizontal plane	If 2 or 3 points are all in the same horizontal plane, it means they are all level or on the same flat surface. One will not be higher or lower that the other
perpendicular	At a right angle. Perfectly straight
vertical	Upright. Perpendicular (at a right angle) to the ground
2-dimensional	Flat. Can only measure two dimensions (usually length and breadth)

angle of elevation	 The angle that a person must look UP to see an object The angle is formed between the horizontal and the line of sight
angle of depression	 The angle that a person must look DOWN to see an object The angle is formed between the horizontal and the line of sight

Term 3, Topic 3, Lesson 1

REVISION OF RIGHT-ANGLED TRIGONOMETRY

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners will have revised: • solving right-angled triangles including word problems with angles of elevation and depression.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson draw the triangle forming three right-angled triangles (point 3).
5. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 10 textbook or items from a Grade 10 test on solving triangles using trigonometry.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
16	171	1	215	Qu's	228	10.1	332		

CONCEPTUAL DEVELOPMENT

INTRODUCTION

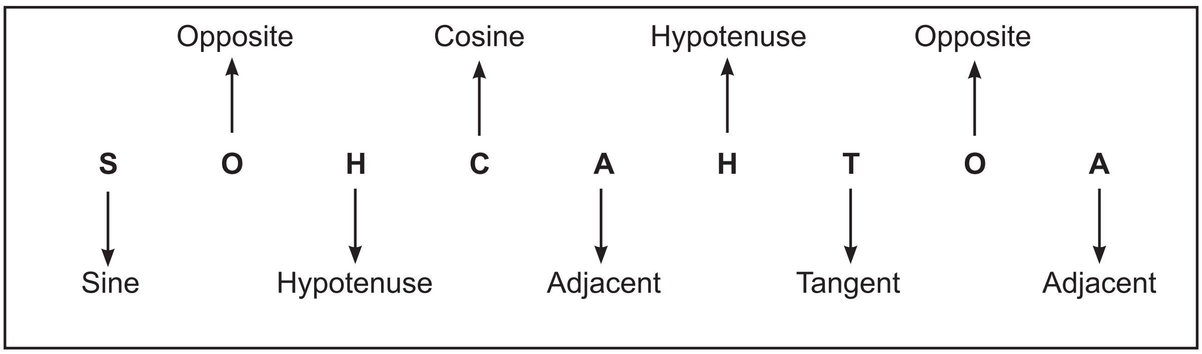
1. Learners have solved right-angled triangles in Grade 10 and again earlier this year. They should not need much practice. As always adjust the lesson to suit the needs of your own learners.
2. Ensure learners are proficient in solving right-angled triangles before moving on.

DIRECT INSTRUCTION

1. Start the lesson by asking learners to tell you the three main trigonometric ratios and which sides in a right-angled triangle are connected to each one.

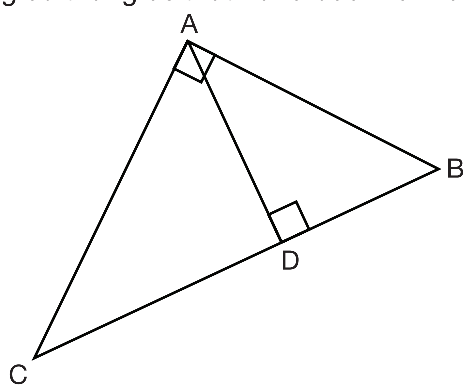
   sine: $\frac{opposite}{hypotenuse}$ cosine: $\frac{adjacent}{hypotenuse}$ tangent: $\frac{opposite}{adjacent}$

2. If no-one mentioned it remind them of the mnemonic they can use to assist them in remembering these:



   Learners should write the mnemonic in their exercise books.

3. Show learners the triangle you have drawn on the board.
   Ask: *Name all the right-angled triangles that have been formed.*



(ΔABC ΔADC ΔADB)

4. Tell learners you are going to call out a trigonometric ratio for a certain angle and they must give you all the ratios possible (one or two, depending on the angle and whether it is in only one right-angled triangle or two). When one learner gives a ratio, ask another learner which triangle they were working in.

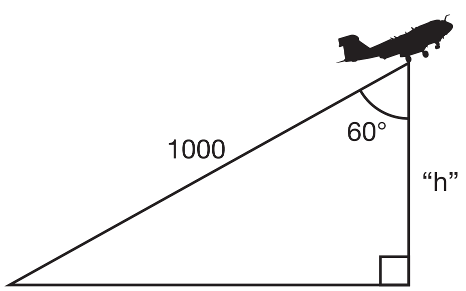
   For example, tan $\hat{C} = \frac{AD}{CD}\ (\Delta ADC)$ OR tan $\hat{C} = \frac{AB}{AD}\ (\Delta ABC)$ $\longrightarrow$ 2 possible ratios

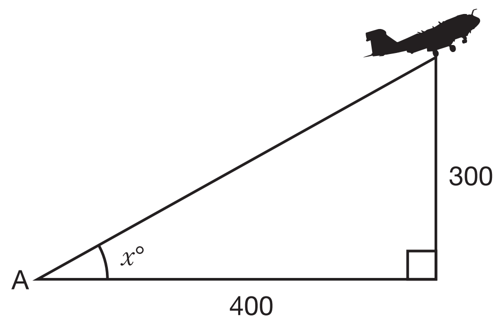
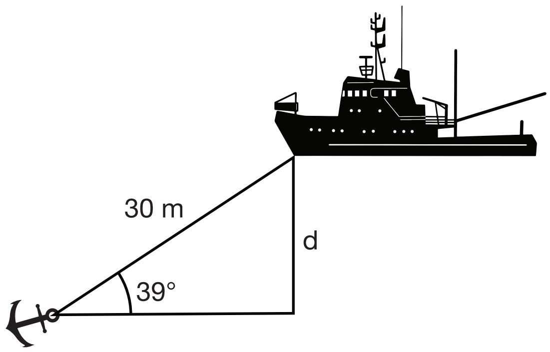
   But, tan $C\hat{A}D = \frac{CD}{AD}$ $\longrightarrow$ only one ratio

5. Use the following table to ask learners questions:

	ΔACD	ΔABC		ΔABD	ΔABC
$\sin C$	$\frac{AD}{AC}$	$\frac{AB}{BC}$	$\sin B$	$\frac{AD}{AB}$	$\frac{AC}{BC}$
$\cos C$	$\frac{CD}{AC}$	$\frac{AC}{BC}$	$\cos B$	$\frac{BD}{AB}$	$\frac{AB}{BC}$
$\tan C$	$\frac{AD}{CD}$	$\frac{AB}{AC}$	$\tan B$	$\frac{AD}{BD}$	$\frac{AC}{AB}$
ΔACD			ΔABD		
$\sin C\hat{A}D$	$\frac{CD}{AC}$		$\sin B\hat{A}D$	$\frac{BD}{AB}$	
$\cos C\hat{A}D$	$\frac{AD}{AC}$		$\cos B\hat{A}D$	$\frac{AD}{AB}$	
$\tan C\hat{A}D$	$\frac{CD}{AD}$		$\tan B\hat{A}D$	$\frac{BD}{AD}$	

6. Do the following worked examples with learners. Learners should write the worked examples in their exercise books.

	Teaching notes	**Solution**
 Find h	Ask: *According to the angle named what is given and what is required?* (hypotenuse and adjacent) Ask: *What trig ratio will be used?* (cosine)	$\cos 60^o = \frac{h}{1000}$ $1000 \cos 60^o = h$ $\therefore h = 500$

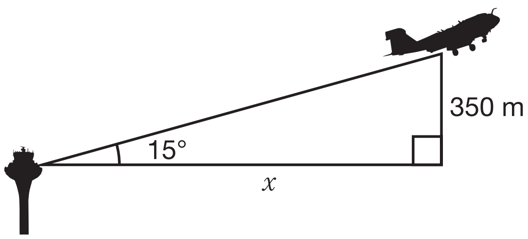
 Find x	Ask: *According to the angle named what is given?* (opposite and adjacent) Ask: *What trig ratio will be used?* (tangent)	$\tan x = \frac{300}{400}$ $\therefore x = 36,87^o$
 Find d	Ask: *According to the angle named what is given and what is required?* (hypotenuse and opposite) Ask: *What trig ratio will be used?* (sine)	$\sin 39^o = \frac{d}{30}$ $30 \sin 39^o = d$ $\therefore d = 18,88\text{m}$

7. Ask: *What is an angle of elevation and an angle of depression?*

   Angle of elevation: The angle that a person must look UP to see an object.
   The angle is formed between the horizontal and the line of sight.

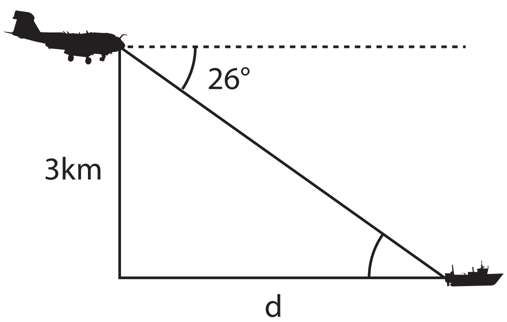
   Angle of depression: The angle that a person must look DOWN to see an object.
   The angle is formed between the horizontal and the line of sight.

8. Tell learners that these are important concepts and they must ensure they understand them clearly. Ask if anyone has any questions before you do two examples using both types of angles. Learners should write the examples in their books.

Example 1	Teaching notes
A control tower operator is looking at an airplane taking off. If he can see the airplane at an angle of elevation of 15° and the airplane is 350m above the control tower, what is the horizontal distance from the control tower to the airplane? 	Tell learners to look at the horizontal line and see how the angle is made by moving UP. Ask: *According to the angle named what is given and what is required?* (opposite and adjacent) Ask: W*hat trig ratio will be used?* (tangent)

Solution:

$$\tan 15^o = \frac{350}{x}$$
$$x \tan 15^o = 350$$
$$x = \frac{350}{\tan 15^o}$$
$$\therefore x = 1306{,}22\text{m}$$

Example 2	Teaching notes
An airplane pilot sights a life raft at a 26° angle of depression. The aeroplane's altitude is 3km. What is the airplane's surface distance d from the raft? 	Tell learners to look at the horizontal line and see how the angle is made by moving DOWN. Remind learners that the horizontal line of sight is parallel to the ground and it should therefore be clear that the angle at the raft (inside the triangle) is also 26° due to alternate angles being equal. Ask: *According to the angle named what is given and what is required?* (opposite and adjacent) Ask: W*hat trig ratio will be used?* (tangent)

Solution:

$$\tan 26^o = \frac{3}{d}$$
$$d \tan 26^o = 3$$
$$d = \frac{3}{\tan 26^o}$$
$$\therefore x = 6{,}15\text{m}$$

9. Ask directed questions so that you can ascertain learners' level of understanding.
   Ask learners if they have any questions.

10. Give learners an exercise to complete on their own.

11. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=3umBUlrUCPQ

https://www.youtube.com/watch?v=e_QPK-pOPqY

(Although these videos do not use metric measurements the explanation is still useful)

Term 3, Topic 3, Lesson 2

THE SINE RULE

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● describe when the sine rule can be used ● solve triangles using the sine rule.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson draw and label triangle ABC.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
17	174	2	216	2	237	10.3 10.4	343 345	6.11	293

CONCEPTUAL DEVELOPMENT

C

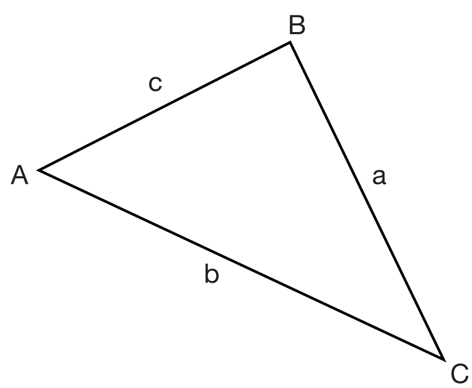
INTRODUCTION

1. The sine rule (or law of sines) is the first of three rules used to solve non-right-angled triangles.
2. Learners need to be proficient in the use of all three rules as they will be used again in Grade 12 trigonometry.

DIRECT INSTRUCTION

1. Start the lesson by pointing out to learners that until now, all the trigonometry they have encountered has been in right-angled triangles only and this may seem as if trigonometry only works in right-angled triangles.
2. Tell learners that this is not the case and you are going to look at the first of two rules that will help us solve non-right-angled triangles. After that you will also look at how to use trigonometry to find the area of a triangle when we don't have enough information to use the formula $\frac{1}{2}b \times \perp h$.
3. The first of the two rules is the sine rule. Refer learners to the triangle drawn on the board. Remind learners how we label the sides of a triangle:

   It is common practice to label the vertices of a triangle with capital letters and the sides with small letters. It is also common practice to label the side opposite angle A with a small a, the side opposite angle B with a small b and the side opposite angle C with a small c
4. Label the sides of the triangle that is drawn on the board:



5. Show learners that the longest side is ALWAYS opposite the largest angle in any triangle and that similarly, the shortest side is ALWAYS opposite the smallest angle. This is useful to remember when checking if an answer looks correct when solving for an angle or the length of a side.

6. Ask learners to copy the drawing of the triangle down in their books under the heading: *The Sine Rule.*

7. Tell learners that this concept will always be used when using the two rules that we are going to learn and in fact the rules themselves are written in such a way that they would only work if the triangle that has just been drawn is used.

8. Write the sine rule on the board:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

9. Learners should notice that when an angle is used it is linked to its matching side (the opposite side).
   Point out that this rule could also be used when the reciprocals are applied:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

10. In general, it is easier to start an equation with the unknown in the numerator's position. In other words, if an angle is unknown, the first version (Point 8) would be used and if a side is unknown, the second version (Point 9) would be used.

11. Learners should write both versions of the rule, along with a note when to use each form of the rule, in their exercise books now.

12. Point out to learners that when they use the sine rule, they only need a pair of sides and angles to form an equation and solve for the unknown.
    For example, they may only need to use:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

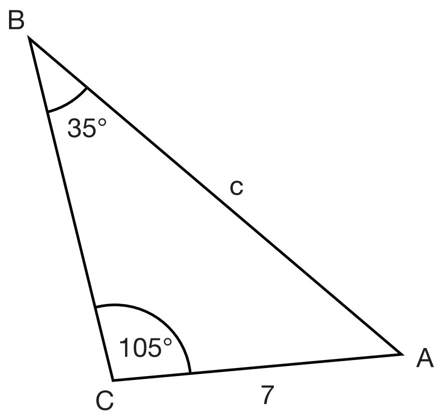
    and, if the triangle is not labeled ABC, they will need to adjust the formula accordingly. For example,

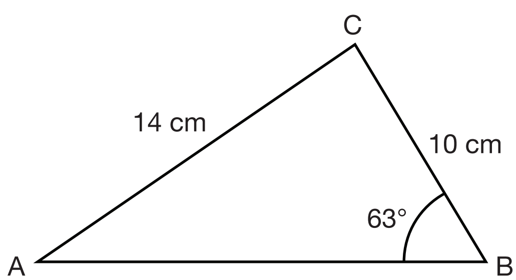
$$\frac{\sin P}{p} = \frac{\sin R}{r}$$

    It is important to remember is that the variables in each fraction MUST be the same as they represent a pair – the angle and its opposite side.

13. Before doing some worked examples, point out to learners that this rule can only work if certain information is supplied. As this is the first rule that they are learning, the exercises that they will do can only be the sine rule. You will explain in more detail in a later lesson how to decide which rule should be used taking the giving information into consideration.

14. Do two worked examples with learners. Example 1 will require finding the length of the will and Example 2 will have an angle as the unknown.

Example 1	**Teaching notes**
 Find AB.	Ask learners to note that a side is required. They should therefore use the version of the rule that has the side in the numerator position. Ask: *What should the equation be?* Note: In the beginning, encourage learners to write down the full formula that they will be using before they substitute known values. As they become more proficient they can leave this step out.
Solution: $\frac{c}{\sin \mathrm{C}} = \frac{b}{\sin \mathrm{B}}$ $\frac{c}{\sin 105^{o}} = \frac{7}{\sin 35^{o}}$ $c \sin 35^{0} = 7 \sin 105^{o}$ $\frac{c \sin 35^{0}}{\sin 35^{0}} = \frac{7 \sin 105^{o}}{\sin 35^{0}}$ $\therefore c = 11{,}79$ units	Remind learners that when two fractions are equal, cross multiplication can be performed to remove the fractions. Use $\frac{1}{2} = \frac{5}{10}$ and show that $(1 \times 10) = (2 \times 5)$ Essentially, this is a 'shortcut' to finding the LCD and multiplying both sides by it. Point out that 105° > 35° $\therefore$ we were expecting our answer to be greater than 7

Example 2	**Teaching notes**
 Find $\hat{\mathrm{A}}$	Ask learners to label sides according to the angle that they are opposite from and to note that an angle is required. Learners should therefore use the version of the rule that has the angle in the numerator position. Ask: *What should the equation be?* Ask: *what will be different in the calculation if an angle is required*? (The 2nd function on the calculator will be used)

Solution:

$$\frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b}$$

$$\frac{\sin \mathrm{A}}{10} = \frac{\sin 63^o}{14}$$

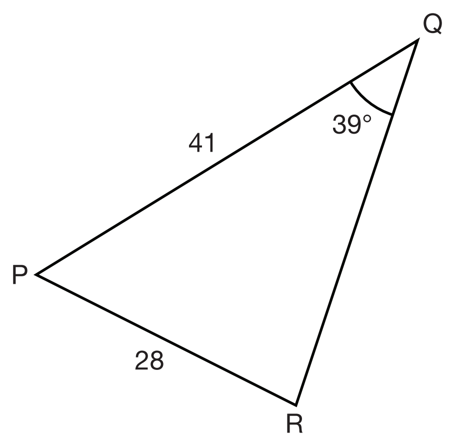
$$14 \sin A = 10 \sin 63^o$$

$$\sin \mathrm{A} = \frac{10 \sin 63^o}{14}$$

$$\therefore \hat{\mathrm{A}} = 39,5^o$$

Point out that 10 < 14, $\therefore$ we were expecting our answer to be less than 63°

15. Ask if anyone has any questions before moving on to the next example which will combine both questions into one and require a knowledge of Grade 8 Geometry.

Example 3	**Teaching notes**
 Solve $\Delta\mathrm{PQR}$	Tell learners to label the sides according to the angle that they are opposite from. Tell learners that 'solve $\Delta\mathrm{PQR}$' is a short way of asking that all missing measurements be found. In this case: QR, $\hat{\mathrm{P}}$ and $\hat{\mathrm{R}}$. Ask: *What can be found first using the sine rule?* ($\hat{\mathrm{R}}$) Ask: *What can be found next and how?* ($\hat{\mathrm{P}}$ using <'s of a Δ) Ask: *How can QR be found?* (Using the sine rule) Remind learners to stop after each answer and check if it makes sense according to other measurements on the diagram.

Solution:

$$\frac{\sin \mathrm{R}}{r} = \frac{\sin \mathrm{Q}}{q}$$
$$\frac{\sin \mathrm{R}}{41} = \frac{\sin 39^o}{28}$$
$$28 \sin \mathrm{R} = 41 \sin 39^o$$
$$\sin \mathrm{R} = \frac{41 \sin 39^o}{28}$$
$$\therefore \hat{\mathrm{R}} = 67,1^o$$
$\therefore \hat{\mathrm{P}} = 73,9^o$ (<'s of Δ
$$\frac{p}{\sin \mathrm{P}} = \frac{q}{\sin \mathrm{Q}}$$
$$\frac{p}{\sin 73,9^o} = \frac{28}{\sin 39^o}$$
$$p \sin 39^o = 28 \sin 73,9^o$$
$$p = \frac{28 \sin 73,9^o}{\sin 39^o}$$
$$p = 42,75 \text{ units}$$

16. Tell learners we will look at examples from previous examinations in a later lesson as most questions from this section of trigonometry combine at least two of the rules within the same question.
17. If you feel your learners need another example, do one or two more from the textbook or any other source.
18. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
19. Give learners an exercise to complete on their own.
20. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=bDPRWJdVzfs
https://www.youtube.com/watch?v=r4YJuhS-1XE
(Tutorials in the sine rule)

https://www.youtube.com/watch?v=natJoNPhkWw
(A light-hearted song about the sine rule)

Term 3, Topic 3, Lesson 3

THE COSINE RULE

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● describe when the cosine rule can be used ● solve triangles using the cosine rule.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 16 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw the first triangle (point 2).
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
18	178	3	218	3	243	10.5	352	6.12	298

FET_Maths_Term3_Gr11.indb 104 2018/06/11 3:08:17 PM

CONCEPTUAL DEVELOPMENT

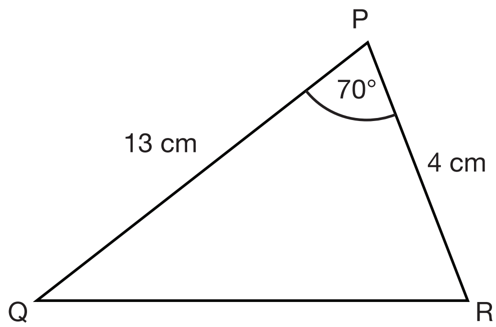
C

INTRODUCTION

1. The cosine rule (or law of cosines) is the second of three rules used to solve non-right-angled triangles.
2. Learners need to be proficient in the use of all three rules as they will be used again in Grade 12 trigonometry.

DIRECT INSTRUCTION

1. Tell learners that we are now going to look at the second rule which will help us to solve non-right-angled triangles. Start by explaining why another rule is required.
2. Use the triangle you have drawn on the board to demonstrate:



   Ask learners to label the sides according to their opposite angles.

   Ask: *Make a statement about* p *(QR) using the sine rule.*

   Ask: *What do you notice?* (There isn't a full pair to find the length of QR).

   Ask: *Make a statement about* $\hat{R}$*.*

   Ask: *What do you notice?* (There isn't a full pair to find the size of $\hat{R}$)

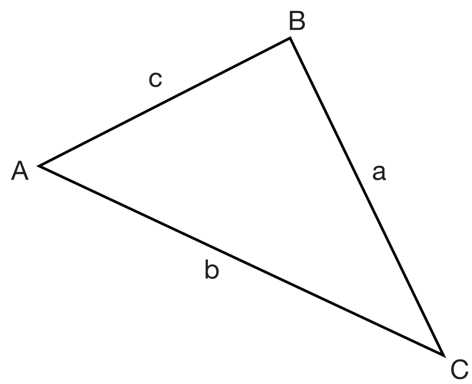
3. Point out that this problem would continue because there are minimum requirements for using the sine rule:

   - You will need to know at least one pair of a side with its opposite angle to use the sine rule.

   Tell learners to look at the triangle again and ask: C*an you see that each piece of information given doesn't have the measurement of the opposite side or angle?*

4. Tell learners that this is the reason a second rule is required. If learners are given either three sides or two sides and the included angle, then the cosine rule will be required as there will not be enough information to use the sine rule.

5. Ask learners to copy the drawing of the triangle down in their books under the heading: The Cosine Rule.



6. Write the cosine rule on the board for learners to copy:

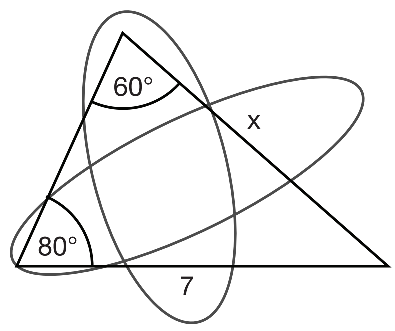
$$a^2 = b^2 + c^2 - 2bc\cos A$$

7. Tell learners that this is always the version of the rule that they will be given. Learners need to notice that the starting variable must always be the same as the ending variable. In other words, the side leading the equation is opposite the angle ending the equation. This should assist learners when the triangle is not labelled ABC and/or or are not the measurements of interest.

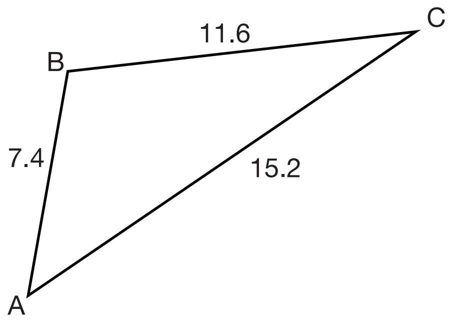
8. If any learners notice that the statement looks like the theorem of Pythagoras at first share the following with them:
   Ask: *What is cos 90° equal to?* (0)
   Therefore, if we were working in a right-angled triangle, what would the cos rule now be?
   ($a^2 = b^2 + c^2 - 2bc\cos A$ as the last term would equal zero).

9. Before doing a few worked examples with learners, summarise what is required to use the cosine rule:
   - 3 sides
   - 2 sides and an included angle.

Encourage learners to check for opposite pairs which will immediately let them know whether the sine rule can be used. One full pair of a known angle and side is required and one pair with the unknown in it is required. If the sine rule can't be used, then it must be the cosine rule. The diagram below shows what learners should look for.



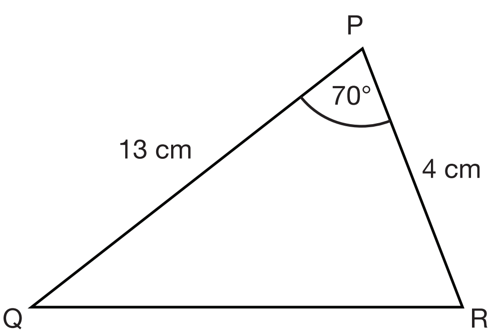
10. Do two worked examples. Example 1 has an angle as the unknown. Example 2 requires finding the length of the side.

Example 1	**Teaching notes:**
 Find $\hat{B}$	Learners should label the sides according to the angle that they are opposite from and to note that they are needing to find $\hat{B}$. This requires a small variation to the formula given. b and $\hat{B}$ must start and end the formula. Ask: *What will be different in the calculation if an angle is required*? (The 2nd function on the calculator will be used).

Solution:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$(15,2)^2 = (11,6)^2 + (7,4)^2 - 2(11,6)(7,4)\cos B$$
$$231,04 = 189,32 - 171,68\cos B$$
$$41,72 = -171,68\cos B$$
$$\frac{41,72}{-171,68} = \cos B$$
$$\therefore \hat{B} = 104,1^o$$

1. A common error learners should be aware of: they cannot subtract 171,68 from 189,32 after substituting and squaring. The $-171,68$ is part of the term with cos B in it.
2. Ask: *Explain why a negative cos ratio gave the answer it did.* (cos is negative in the 2nd and 3rd quadrants and should therefore produce an angle lying in one of those quadrants).

Example 2	**Teaching notes:**
 Find QR	Learners should label the sides according to the angle that they are opposite from and therefore to note that they are needing to find p. This will require stating the formula by starting with P and ending with cos P.

Solution:

$p^2 = q^2 + r^2 - 2qr\cos \mathrm{P}$

$p^2 = (4)^2 + (13)^2 - 2(4)(13)\cos 70^\circ$

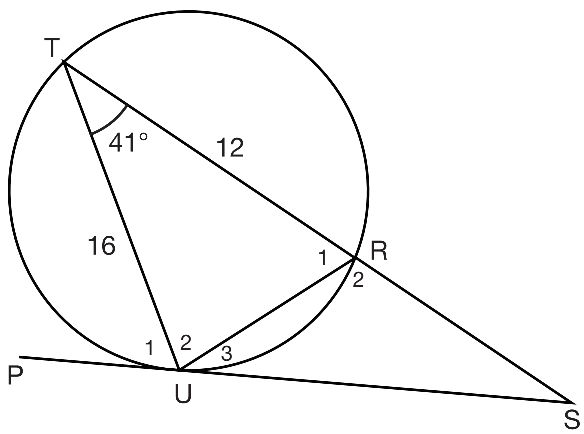
$p^2 = 149,429....$

$\therefore p = 12,22 \text{ cm}$

Learners should leave the full answer for p^2 in their calculator before square rooting to ensure as accurate an answer as possible.

11. Ask if anyone has any questions before moving on to the next example which will combine the use of both rules. Tell learners that if there is enough information to use either one of the rules, the sine rule is usually less work and therefore the quickest to use.

The diagram for the example below is available in the Resource Pack. Resource 16.

Example	**Teaching notes**
TRS is a secant of the circle, and SU is a tangent at U. TU = 16cm, TR = 12cm and $\hat{\mathrm{T}} = 41^\circ$.  Calculate the: a) length of UR, correct to two decimal places.	Learners should read through the information carefully and check it with the diagram. Remind learners to label the sides according to the angle that they are opposite from. Ask: *What other knowledge is expected in this question?* (Euclidean geometry) Ask: *do we know a side and its opposite angle?* (no) *What rule will we use?* (cosine) *What rule will we use?* (sine)
b) size of $\hat{\mathrm{U}}_2$.	Ask: *Do we know a side and its opposite angle?* (Yes) *What rule will we use?* (sine)
c) length of secant TRS. EC 2016	Ask: *what do we really need to find?* (RS) *do we know a side and its opposite angle?* (no – but we can use geometry to find a full pair)

Solution

a) $t^2 = u^2 + r^2 - 2ur\cos\ \text{T}$

$t^2 = (16)^2 + (12)^2 - 2(16)(12)\cos 41^o$

$t^2 = 110,191\ldots$

$\therefore t = 10,50 \text{ cm}$

$\therefore \text{RU} = 10,50 \text{ cm}$

b) $\dfrac{\sin \text{U}}{12} = \dfrac{\sin 41^o}{10,5}$

$10,5\sin \text{U} = 12\sin 41^o$

$\sin \text{U} = \dfrac{12\sin 41^o}{10,5}$

$\therefore \hat{\text{U}} = 48,6^o$

$\therefore \hat{\text{U}}_2 = 48,6^o$

c) $\hat{\text{U}}_3 = 41^o$ (tan-chord)

$\hat{\text{R}}_2 = 89,6^o$ (ext < Δ)

$\therefore \hat{\text{S}} = 49,4^o$ (<'s of Δ)

$\dfrac{u}{\sin 41^o} = \dfrac{10,5}{\sin 49,4^o}$

$u = \dfrac{10,5\sin 41^o}{\sin 49,4^o}$

$\therefore u = 9,07$

$\therefore \text{RS} = 9,07$

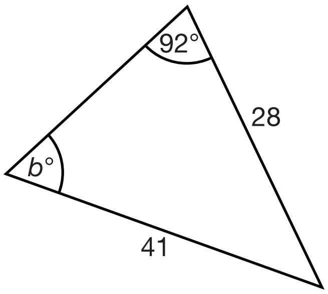
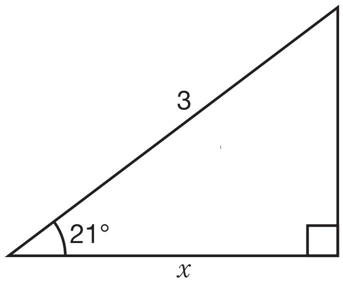
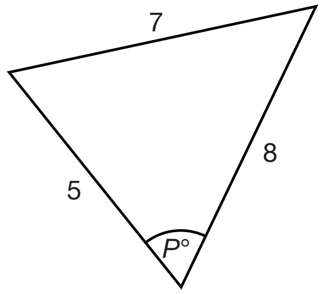
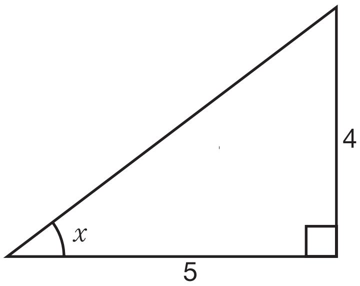
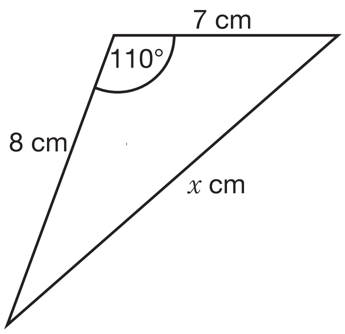
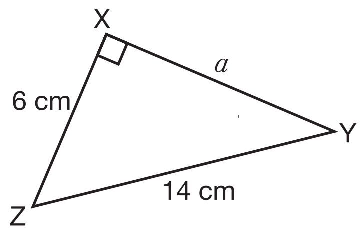
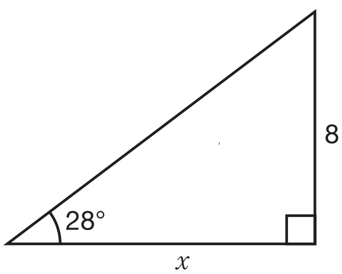
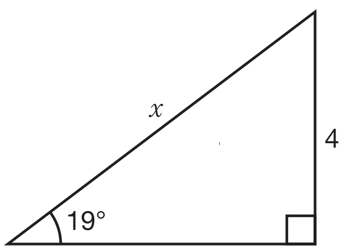
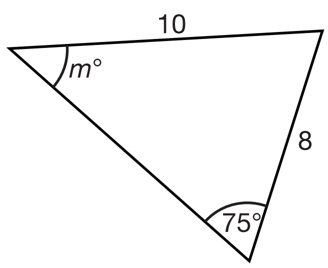
$\therefore \text{TRS} = 21,07$

Note: There is more than one way of answering this question.
For example, c) could have been done by working in ΔTUS and using the sine rule.

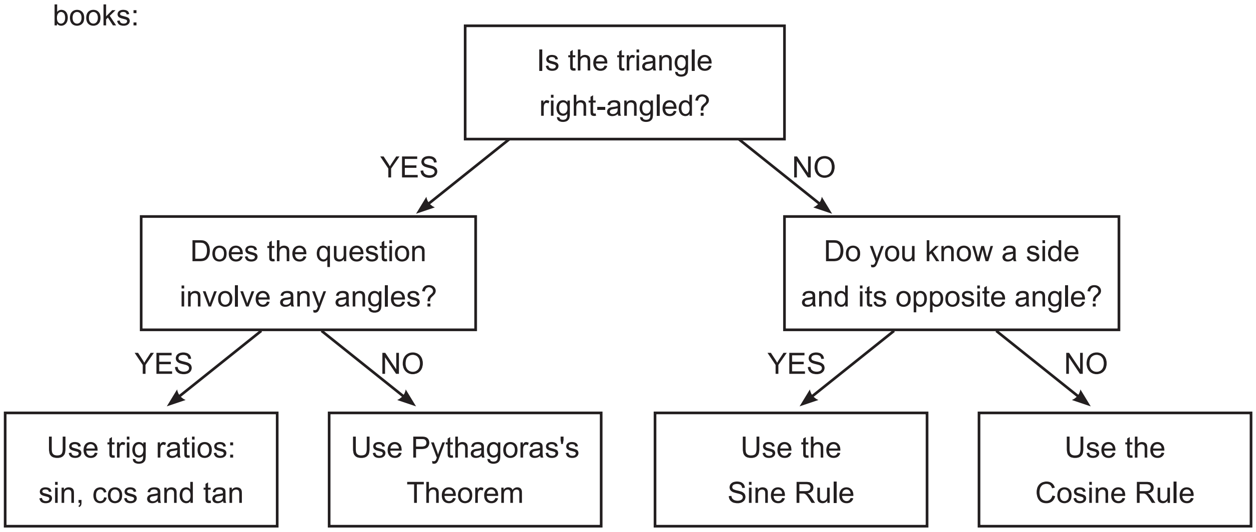
Once the sine and cosine rules have been learned, learners often want to use them even if it is not essential to do so. In other words, they may use the rules in a right-angled triangle when they could have just used right-angled trigonometry.
Although this is not a problem mathematically, reverting to right-angled trigonometry is preferred.

12. Draw the following triangles on the board. Quiz the learners on what they would use to find the unknown.

    Sine rule, cosine rule, sin/cos/tan (right-angled trigonometry) or Pythagoras?

		
sine rule	cos $\left(\frac{adj}{hyp}\right)$	cosine rule
		
tan $\left(\frac{opp}{adj}\right)$	cosine rule	Pythagoras
		
tan $\left(\frac{opp}{adj}\right)$	sin$\left(\frac{opp}{hyp}\right)$	sine rule

Draw (and discuss as you are drawing) the following flow chart for learners to copy into their books:



13. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

14. Give learners an exercise to complete on their own.

15. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=iEWSqAk3hTw
(When to use the cosine rule)

https://www.youtube.com/watch?v=VKQfiJaEJvc
https://www.youtube.com/watch?v=qGRiZ6jsBJo
(Examples)

https://www.youtube.com/watch?v=-wsf88ELFkk
(A light-hearted rap song about the cosine rule)

Term 3, Topic 3, Lesson 4

THE AREA RULE

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● describe what is required to find the area of a triangle ● find the area of triangles using the area rule.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. Draw the two triangles for point 2.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
19	181	4	220	1	232	10.2	336	6.10	285

CONCEPTUAL DEVELOPMENT

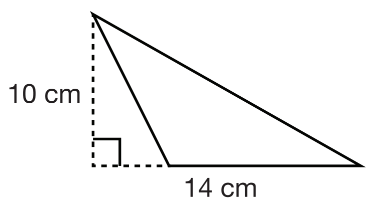
C

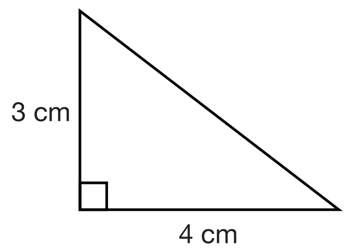
INTRODUCTION

1. The area rule is the final rule regarding non-right-angled triangles. It is used to find area when the base and/or perpendicular height are unknown.
2. Learners need to be proficient in the use of all three rules as they will be used again in Grade 12 Trigonometry.

DIRECT INSTRUCTION

1. Start the lesson by asking: *How do we find the area of a triangle?*
   ($\frac{1}{2}$ base $\times \perp$ height).
   Ask learners to find the area of the following two triangles:

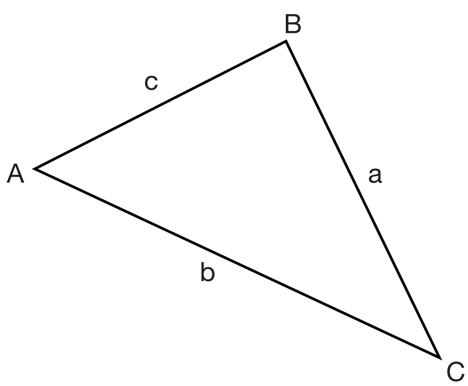




Area $= \frac{1}{2}$ base $\times \perp$ height
$= \frac{1}{2}(14)(10)\,\text{m}^2$
$= 70\text{m}^2$

Area $= \frac{1}{2}$ base $\times \perp$ height
$= \frac{1}{2}(4)(3)\,\text{cm}^2$
$= 6\text{cm}^2$

2. Tell learners that they can be asked to find area of a triangle where the base and/or the perpendicular height is not given, and that trigonometry can be used.
3. Ask learners to copy the drawing of the triangle down in their books under the heading: <u>The Area Rule.</u>



4. Write the area rule on the board for learners to copy down:

$$\text{Area} = \frac{1}{2}ab\ \sin \text{C}$$

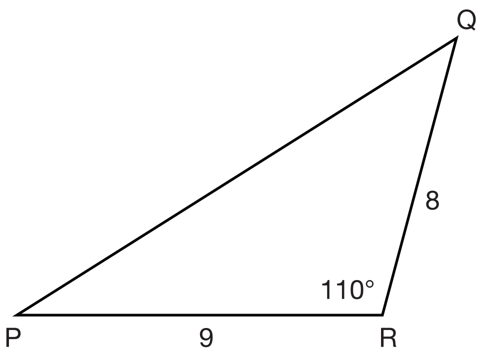
5. Get learners to focus their attention on a, b and $\hat{\text{C}}$
   Ask: *What is the relationship between these two sides and this angle?*
   (The angle is between the two sides).

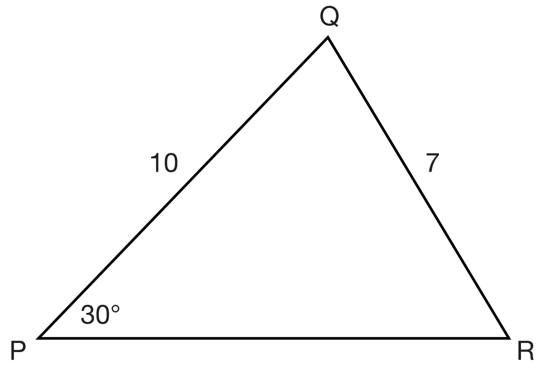
6. Point out that two sides and the included angle are always necessary to use the area rule.

7. Ask: *Which other combinations could be used in this triangle to find the area?*
   a, c and $\hat{\text{B}}$ or b, c and $\hat{\text{A}}$

8. Tell learners that it is quite common to be asked an area question where the necessary information is not given and either the sine or cosine rule need to be used to find the information required.

9. Do the following two worked examples. Learners should write them in their exercise books.

Example 1	**Teaching notes**
Find the area of ΔPQR 	Remind learners to label the sides according to the angle that they are opposite from. This is a straightforward question where the information required (two sides and an included angle) is given. The values just need to be substituted into the formula. Learners need to be aware that the need to change the formula to suit the vertices of the given triangle.
Solution: Area $= \frac{1}{2}pq\ \sin\ \text{R}$ Area $= \frac{1}{2}(8)(9)\sin 110^{o}$ Area $= 33,83\ \text{units}^2$	

Example 2	**Teaching notes**
Find the area of $\Delta\mathrm{PQR}$ 	Remind learners to label the sides according to the angle that they are opposite from. Ask: *Do we have the required information?* (No, we need $\hat{\mathrm{Q}}$ so that two sides and an included angle will be available) Ask: *Do we have a complete pair of an opposite side and angle?* (Yes) Ask: *Can we use the sine rule to find* $\hat{\mathrm{Q}}$*?* (No, but we can use it to find $\hat{\mathrm{R}}$ then use <'s of a triangle to find $\hat{\mathrm{Q}}$).
Solution: $\frac{\sin \mathrm{R}}{10}=\frac{\sin 30^{o}}{7}$ $7\sin\mathrm{R}=10\sin 30^{o}$ $\sin\mathrm{R}=\frac{10\sin 30^{o}}{7}$ $\therefore \hat{\mathrm{R}}=45{,}6^{o}$ $\therefore \hat{\mathrm{Q}}=104{,}4^{o}$ (<'s of Δ) Area $=\frac{1}{2}pr\ \sin\ Q$ Area $=\frac{1}{2}(7)(10)\sin 104{,}4^{o}$ Area $=33{,}90\ \mathrm{units}^{2}$	

10. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

11. Give learners an exercise to complete on their own.

12. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=jVYjb0fgKaA

Term 3, Topic 3, Lesson 5

PROOFS OF THE 3 RULES AND THE AMBIGUOUS CASE OF THE SINE RULE

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● prove the sine, cosine and area rules.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson draw the diagram for the proof of the sine rule.
5. In the textbooks there are no exercises specifically testing knowledge of the proofs.

CONCEPTUAL DEVELOPMENT

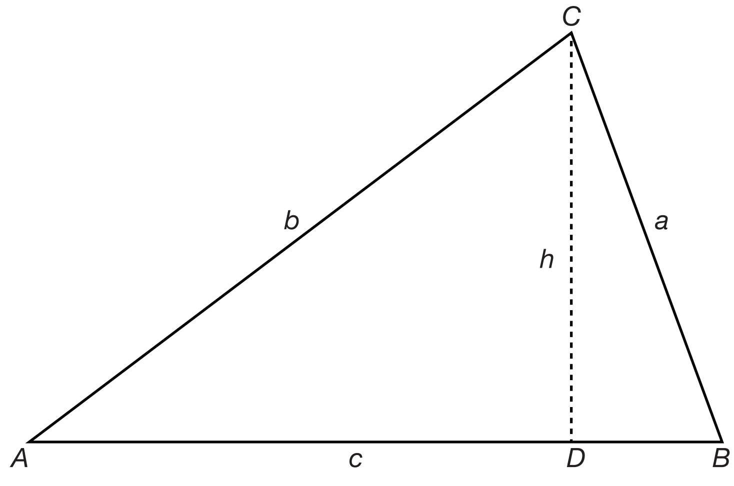
C

INTRODUCTION

1. This lesson has been placed after all three rules have been learned and practiced.
2. Use this lesson as an opportunity to consolidate learners' understanding of the three rules. Tell learners that knowing and understanding the proofs can assist them in making more sense of the rules.
3. The ambiguous case of the sine rule was not covered in the lesson on the sine rule. Experience has shown that most learners need to spend time on the rule itself before alerting them to the 'problems' that could occur when dealing with the sine rule. The ambiguous case of the sine rule has been covered in the form of an investigation which learners can do with you in class.

DIRECT INSTRUCTION

1. Start the lesson by telling learners that we are now going to prove the three rules that have just been covered.
2. Tell learners to write the heading 'Proof of the Sine Rule' in their exercise books and copy the diagram:

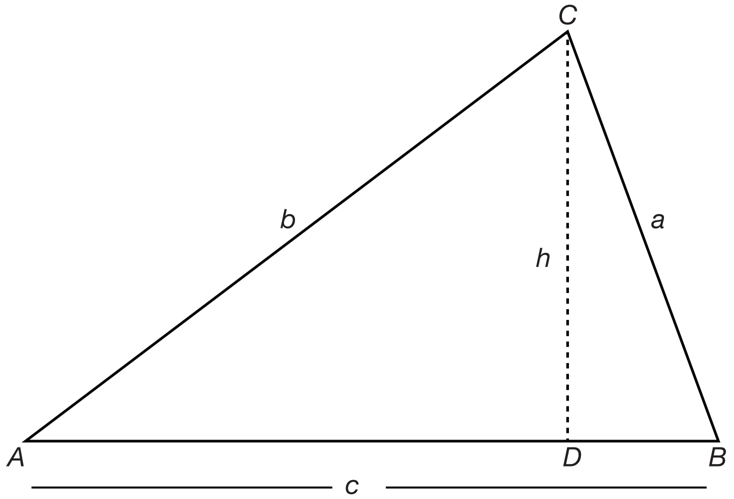


3. Go through the proof orally before writing it up with learners:
   a) Get learners to focus on ΔACD. Ask: *What is sin A equal to?* $\left(\frac{h}{b}\right)$
   b) Get learners to focus on ΔBCD. Ask: *What is sin B equal to?* $\left(\frac{h}{a}\right)$
   c) Point out to learners that h appears in both answers and that we will work with what we have found to make h the subject of the formula. This way we will find what else is equal.

4. Write up the proof and discuss it as you do each step:

In ΔACD: $\sin A = \frac{h}{b}$ $\therefore h = b\sin A$	In ΔBCD : $\sin B = \frac{h}{a}$ $\therefore h = a\sin B$
$\therefore b\sin A = a\sin B$ (÷ both sides by ab) $\frac{b\sin A}{ab} = \frac{a\sin B}{ab}$ $\therefore \frac{\sin A}{a} = \frac{\sin B}{b}$ Similarly, if the altitude was drawn from B, $\frac{\sin A}{a} = \frac{\sin C}{c}$ and if the altitude was drawn from A $\frac{\sin C}{C} = \frac{\sin B}{b}$	

5. Tell learners to write the heading: '<u>Proof of the Cosine Rule</u>' in their exercise books and to copy this diagram:



6. Go through the proof orally before writing it up with learners.

a) Get learners to focus on ΔACD. Ask: What is $\sin A$ equal to? $\left(\frac{h}{b}\right)$

b) Get learners to focus on ΔACD again. Ask: What is cos A equal to? $\left(\frac{AD}{b}\right)$

c) Making these statements allows us to make h and AD the subject of the formula.

d) Get learners to focus on ΔBCD. Write BD in terms of c and AD, then make a theorem of Pythagoras statement.

e) Using what was found in (1) and (2), substitutions can be made into the theorem of Pythagoras statement. This statement will be simplified.

7. Write up the proof and discuss it as you do each step:

In ΔACD:

$\sin A = \frac{h}{b}$ and $\cos A = \frac{AD}{b}$

$\therefore h = b\sin A$ and $AD = b\cos A$

but, $BD = c - AD = c - b\cos A$

In ΔBCD:

$a^2 = BD^2 + h^2$

$\therefore a^2 = (c - b\cos A)^2 + (b\sin A)^2$

$a^2 = c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A$

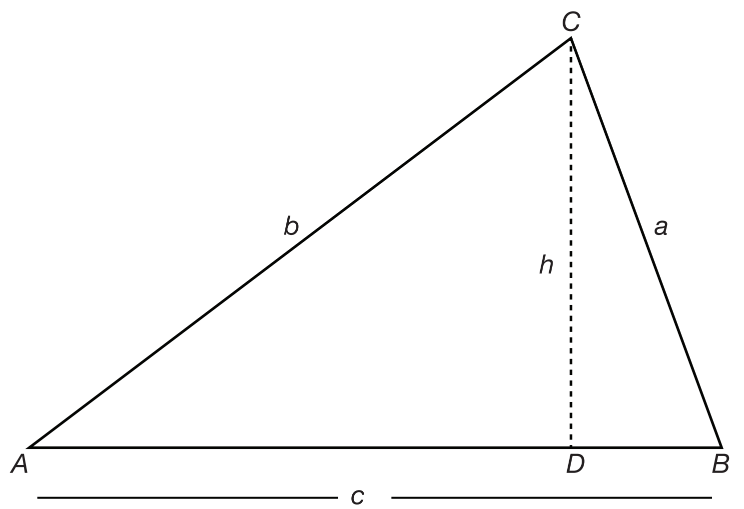
$a^2 = c^2 - 2bc\cos A + b^2(\cos^2 A + \sin^2 A)$

$a^2 = c^2 - 2bc\cos A + b^2(1)$

$a^2 = c^2 + b^2 - 2bc\cos A$

Ask: *How could we simplify the last two terms?*
(Take out a HCF b^2)
Ask: *What would be left over?*
$(\cos^2 A + \sin^2 A)$
Ask: *What does that equal?* (1)

8. Tell learners to write the heading: <u>'Proof of the Area Rule'</u> in their exercise books and copy the diagram:



9. Go through the proof orally before writing it up with learners.

   a) Ask: *How would you find the area of* ΔABC?
   $(\frac{1}{2}b \times \perp ht = \frac{1}{2}ch)$

   b) Get learners to focus on ΔACD. Ask: *What is sin A equal to?* $\left(\frac{h}{b}\right)$

   c) Using statement b) we can make h the subject of the formula and substitute it for h in the area formula.

10. Write up the proof and discuss it as you do each step:

Area of $\Delta\text{ABC} = \frac{1}{2} b \times \perp ht$

$= \frac{1}{2} ch$

In ΔACD:

$\sin \text{ A} = \frac{h}{b}$

$\therefore h = b \sin \text{A}$

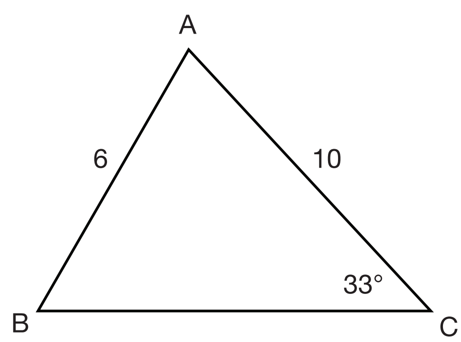
Area of $\Delta\text{ABC} = \frac{1}{2} ch$

$= \frac{1}{2} bc \sin \text{A}$

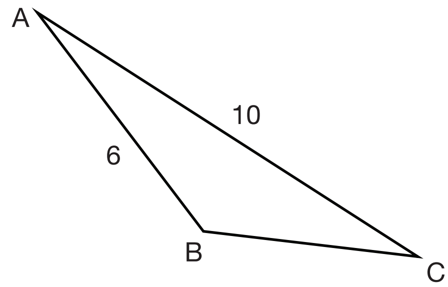
11. Ask if anyone has any questions.

12. Tell learners that you are going to show them the ambiguous case of the sine rule. This means that in certain situations there can be two possible solutions.

13. Go through each step with learners. Tell learners that this situation can only occur if two sides and an angle are given but the angle is not included.

Example		
1.	For ΔABC find $\hat{\text{B}}$ if $\hat{\text{C}} = 33°$, $b = 10$ and $c = 6$ Draw a sketch.	
2.	Use the sine rule to find $\hat{\text{B}}$	$\frac{\sin \text{B}}{10} = \frac{\sin 33°}{6}$ $\sin \text{B} = 0,9077\ldots$ $\therefore \hat{\text{B}} = 65,2°$
3.	We have found one value for angle B, but we need to see if there's another possible value. To do this, we'll subtract the angle from 180°.	180° − 65,2° = 114,8°

<table>
<tr><td>4.</td><td>To see if this is a valid answer, we must add it to the other existing angle. If their sum is less than 180°, we know a triangle can exist. If, however, it is over 180°, then it is not a valid answer (because the three angles of a triangle must add up to 180°).</td><td>$114{,}8° + 33° = 147{,}8°$

Another triangle could exist.</td></tr>
<tr><td>5.</td><td>Let’s see why:</td><td>Could the triangle look like this?

</td></tr>
<tr><td>6.</td><td colspan="2">SO, remember that if you are given two sides and one angle of a triangle (SSA), you don’t have enough information to know exactly what that triangle looks like.
The reason for this possibility:
This can occur because $\sin\theta = \sin(180^o - \theta)$ and when inverse sine is used to find an angle, the calculator is technically only giving the reference angle.</td></tr>
</table>

14. Ask directed questions so that you can ascertain learners’ level of understanding.
    Ask learners if they have any questions.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=7jMMS6geVmk
(Proof of cosine rule – slightly different to the one in the lesson)

https://www.youtube.com/watch?v=Zh2JfMU-sVE
(Proof of sine rule)

https://www.youtube.com/watch?v=P6H8KIWOFEA
(Proof of area rule)

https://www.youtube.com/watch?v=S1oDtGHC2iA
(Ambiguous case of sine rule)

2 DIMENSIONAL QUESTIONS INVOLVING ALL 3 RULES

Suggested lesson duration: 2 hours

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: • solve 2-dimensional problems involving the use of the sine, cosine and area rules.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resources 17 and 18 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw the diagram for the first example.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
20	184	6 7 8 9	223 226 228 230	4 5	246 248	10.6 10.7	356 362	6.13	299

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson is made up of several fully worked examples from past examinations.
2. Most questions combine at least two of the concepts covered in this topic which include, solving right-angled triangles, using the sine and cosine rules to solve non-right-angled triangles and using the area rule to find the area of a triangle.
3. All the questions involve 2-dimensional depictions.

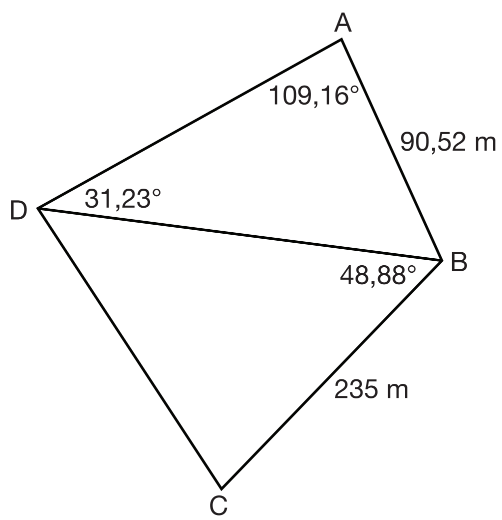
DIRECT INSTRUCTION

1. Tell learners that this lesson is going to be spent doing questions from previous examinations which will allow us to consolidate the rules learned and be exposed to the types of questions to expect.
2. Before starting the examples ask learners to remind you what the following mean:
   - Horizontal plane
   - Angle of elevation
   - Angle of depression

   (Although these were discussed in the first lesson, learners may need a reminder).

3. Do the worked examples with learners. Learners should write the worked examples in their exercise books.

The diagrams in the examples are provided in the Resource Pack. The diagrams have been enlarged for your convenience. Resources 17 and 18.

Example 1	**Teaching notes**
Quadrilateral ABCD is drawn with BC = 235m and AB = 90,52m. It is also given that $\text{A}\hat{\text{D}}\text{B} = 31,23^o$; $\text{D}\hat{\text{A}}\text{B} = 109,16^o$ and $\text{C}\hat{\text{B}}\text{D} = 48,88^o$  Determine the length of: a) BD	Remind learners to label the sides according to the angle that they are opposite from. Tell learners to confirm that all the information from the question has been transferred to the diagram. Ask: *do we know a side and its opposite angle?* (yes) *What rule will we use?* (sine)
Determine the length of: b) CD NSC NOV 2016	Ask: *Do we know a side and its opposite angle?* (no) Ask: *What rule will we use?* (cosine)

Solution:

a) $\frac{\text{BD}}{\sin 109,16^o} = \frac{90,52}{\sin 31,23^o}$

$\text{BD} \sin 31,23^o = 90,52 \sin 109,16^o$

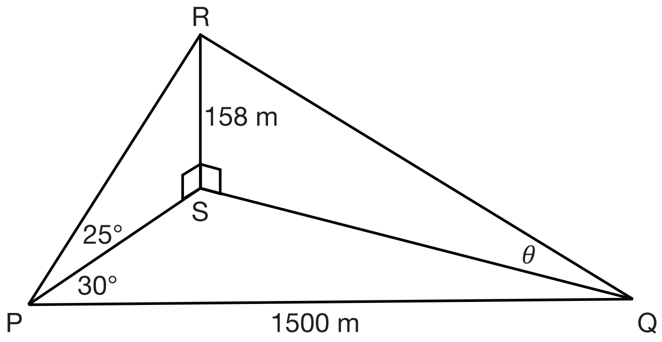
$\text{BD} = \frac{90,52 \sin 109,16^o}{\sin 31,23^o}$

$\therefore \text{BD} = 164,92 \text{ m}$

b) $\text{CD}^2 = (164,92)^2 + (235)^2 - 2(164,92)(235)\cos 48,88^o$

$\text{CD}^2 = 31448,48739$

$\therefore \text{CD} = 177,34 \text{ m}$

Example 2	Teaching notes
In the diagram below, PQ is a straight line 1500m long. RS is a vertical tower 158m high with P, Q and S points in the same horizontal plane. The angles of elevation of R from P and Q are 25° and θ. $\text{S}\hat{\text{P}}\text{Q} = 30^o$. 	Remind learners to label the sides according to the angle that they are opposite from in the non-right-angled triangle.
a) Determine the length of PS.	Ask: W*hich triangle must we work in?* (ΔPRS) Ask: W*hich rule will we use?* (None – use right-angled trigonometry. opp/adj $\therefore$ use tan).
b) Determine the length of SQ.	Ask: *Which triangle must we work in?* (ΔPSQ) Ask: *Which rule will we use?* (cosine – we don't have a complete pair of an opposite side and angle OR we have two sides and an included angle).

c) Hence, find the value of θ.	Ask: W*hich triangle must we work in?* (ΔRSQ) Ask: *Which rule will we use?* (None – use right-angled trigonometry. opp/adj $\therefore$ use tan).
d) Determine the area of ΔSPQ. EC 2015	Ask: D*o we have two sides and an included angle?* (Yes – use formula and substitute)

Solution:

a) $\tan 25^o = \frac{158}{PS}$

$PS \tan 25^o = 158$

$PS = \frac{158}{\tan 25^o}$

$\therefore PS = 338,83$ m

b) $SQ^2 = (338,83)^2 + (1500)^2 - 2(338,83)(1500)\cos 30^o$

$SQ^2 = 1484499,60\ldots$

$\therefore SQ = 1218,4$ m

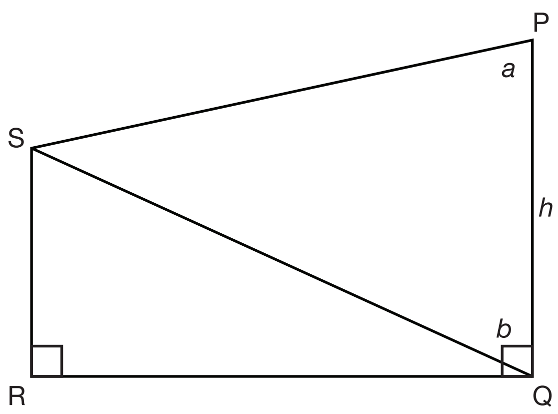
c) $\tan\theta = \frac{158}{1218,4}$

$\therefore \theta = 7,39^o$

d) Area $= \frac{1}{2}(1500)(338,83)\sin 30^o$

Area $= 127061,25 m^2$

Stop and give learners the opportunity to practice some questions from their textbooks that involve actual values. If the textbook you use doesn't have two separate exercises (values vs variables), then give learners only the questions from the exercise that deal with actual values. The rest can be completed after the following example has been done.

Example 3	**Teaching notes**
In the figure below, , $\text{S}\hat{\text{P}}\text{Q} = a$, $\text{P}\hat{\text{Q}}\text{S} = b$ and PQ $= h$. PQ and SR are perpendicular to RQ.  	Remind learners to label the sides according to the angle that they are opposite from. Reassure learners that this question uses all the knowledge that they already have and that the only difference being that their answers will not be actual values.
a) Determine the distance of SQ in terms of a, b and h.	Ask: *Which triangle must we work in?* (ΔPSQ) Point out that although we don't have a complete pair of an opposite side and angle, we do have two angles and can therefore use angles of a triangle to find the 3rd angle. If learners are struggling with the idea of using variables and are not able to find what is required (as in an actual value) advise them to ask themselves what they would do IF all the variables given were actual values. Ask: *Which rule will we use?* (sine rule) Once this has been written down, the expression will need manipulating to make SQ the subject of the formula. When SQ has been found, learners should fill it in on the diagram.
b) Hence, show that $\text{RS} = \dfrac{h \ \sin a \cos b}{\sin(a+b\sin(a+b)}$ EXEMPLAR 2013	Ask: *Which triangle must we work in?* (ΔSRQ) Ask: *Do we have enough information?* (No – we only have a right angle and SQ) Ask: *Can we find another angle?* (Yes - $\text{S}\hat{\text{Q}}\text{R}$ because $\text{R}\hat{\text{Q}}\text{P} = 90^{\circ}$) Once this has been established, a statement will be made regarding RS and manipulation of the expression will be required.

Solution:

a) $\text{P}\hat{\text{S}}\text{Q} = 180^o - (a+b)$

$$\frac{\text{SQ}}{\sin a} = \frac{h}{\sin(180-(a+b))}$$

$$\frac{\text{SQ}}{\sin a} = \frac{h}{\sin(a+b)}$$

$$\text{SQ} = \frac{h\sin a}{\sin(a+b)}$$

> After the first step of the sine rule, ask: How do we reduce angles: (180°−) is in quadrant …. where sine is ….. therefore, we can reduce the angle to…?

a)

$$\text{S}\hat{\text{Q}}\text{R} = 90^o - b$$

$$\sin\ \text{S}\hat{\text{Q}}\text{R} = \frac{\text{RS}}{\text{SQ}}$$

$$\sin(90^\circ - b) = \frac{\text{RS}}{\frac{h\sin a}{\sin(a+b)}}$$

$$\cos\ b = \frac{\text{RS}}{\frac{h\sin a}{\sin(a+b)}}$$

$$\cos\ b.\frac{h\sin a}{\sin(a+b)} = \text{RS}$$

$$\therefore \frac{h\sin a.\cos b.}{\sin(a+b)} = \text{RS}$$

> After the 2nd step (substitution), Ask: *How do we reduce angles in the form 90°−?* (complementary angles)

4. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
5. Give learners an exercise to complete on their own.
6. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=CPjB_z7PZt0
https://www.youtube.com/watch?v=j3VLbjsWdHo
(sine or cosine rule?)

https://www.youtube.com/watch?v=p1ZnrdxNe4I
(Architecture: Law of Sines and Law of Cosines - Why use them?)

Term 3, Topic 3, Lesson 7

REVISION AND CONSOLIDATION

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners will have consolidated and revised the: ● sine rule ● cosine rule ● area rule.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
		Rev	232	Qu’s	250	Rev	364	6.14 (13-15)	302

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to revise what they have learned in this section. Spend time pointing out issues that you know are important as well as problems that you encountered from your own learners during the topic.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

Ask learners to do the revision exercise from their textbook. If you have an additional worksheet or a past test paper, this would also be an excellent way for them to consolidate what they have learned. It would also give them another opportunity of knowing what to expect in an assessment.

Term 3, Topic 4, TOPIC OVERVIEW

FINANCE, GROWTH AND DECAY

TOPIC OVERVIEW

A

- This topic is the fourth of five topics in Term 3.
- This topic runs for two weeks (9 hours).
- It is presented over five lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total 9 hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Finance, Growth and Decay counts 10% of the final Paper 1 examination.
- This topic covers depreciation, nominal and effective interest and different periods of compound growth and decay.
- An understanding of financial matters is an important life skill required by all learners. This is a learned skill which is not taught by all parents – this makes the teacher's role at school even more important.

Breakdown of topic into 5 lessons:

	Lesson title	Suggested time (hours)		Lesson title	Suggested time (hours)
1	Revision	1,5	4	Nominal and Effective interest rates	1,5
2	Depreciation	2	5	Revision and Consolidation	1
3	Different periods of compounding growth and decay	3			

B

SEQUENTIAL TABLE

GRADE 10 and Senior phase	GRADE 11	GRADE 12
LOOKING BACK	**CURRENT**	**LOOKING FORWARD**

• Use the simple and compound growth formulae to solve problems
These problems must include:
• hire purchase
• inflation
• population growth
• exchange rates (implications of fluctuating rates).
	• Use simple and compound decay formulae to solve problems, including:
	  ◦ Straight line depreci-ation
	  ◦ Reducing balance depreciation
	  ◦ Nominal and effective interest rates
	• The effect of different periods of compounding growth and decay.
		• Solve problems involving present and future value annuities
		• Calculate the value of *n* (time period) using logarithms
		• Critically analyse different loan options.

C

WHAT THE NSC DIAGNOSTIC REPORTS TELL US

According to **NSC Diagnostic Reports** there are several issues pertaining to Finance, Growth and Decay.

These include:

- a lack of understanding of the term *depreciation*
- finance needs to be taught with more insight – it is not merely a substitution of values into a formula
- basic algebraic skills, such as multiplication and exponential laws, need to be taught properly then reinforced throughout the FET phase
- learners should be encouraged to check whether their solutions seem ‘reasonable’
- rounding off must only be done at the end of a process – not in the middle of a problem.

It is important that you, as the teacher, keep these issues in mind when teaching this section.

While teaching Financial mathematics, it is important to make it real for the learners.
Where possible, use examples from daily life which make sense to learners.

ASSESSMENT OF THE TOPIC

D

- CAPS formal assessment requirements for Term 3:
  - Two tests
- Two tests, with memorandum, are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of real life situations that require, for example, a calculation of interest with an understanding of nominal and effective interest or calculating an item's worth taking inflation or depreciation into account.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

MATHEMATICAL VOCABULARY

E

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	Explanation
principal amount	The initial or capital sum of money. This amount can represent a borrowed or invested amount of money
profit	Money made when income exceeds (is bigger than) expenditure (You make more money than you spend)
loss	Money lost when expenditure exceeds income (You spend more than you made)
discount	Paying less than the usual price A percentage of the original price is taken off
budget	A plan to manage money
loan	Borrowing of money
interest	The extra money paid back after taking a loan
interest rate	The percentage of interest charged (loan) or received (investment) Rate is often quoted per annum (year)
simple interest	The interest on a loan that is calculated on a yearly basis Interest on a loan amount is charged with only the principal amount taken into consideration
instalment	A sum of money due as one of several equal payments for something, spread over an agreed period of time

exchange rate	The rate of one country's money against another country's money
VAT	Value Added Tax (VAT) s paid on goods or services In South Africa VAT is 15% of the price
hire purchase	When an item is purchased but only a deposit is paid and then the item is paid off monthly over a specified period of time. The interest charged is always simple interest. Officially, the goods bought are being hired until the payments are complete
deposit	A sum of money paid as a first instalment on an item with the understanding that the balance will be paid at a later stage
compound interest	Interest is calculated not only on the principal amount but will also include any accumulated interest that has been added at certain time intervals
inflation	A general increase in prices and fall in the purchasing value of money. Inflation is always calculated using compound interest
fixed deposit	A single deposit invested for a certain period of time at a fixed interest rate
depreciation	A reduction in the value of an asset over time, particularly due to wear and tear (use)
nominal interest	The interest rate before taking inflation into account Nominal can also refer to the advertised or stated interest rate on a loan, without taking into account any fees or compounding of interest
effective interest	The interest rate that is actually earned or paid on an investment, loan or other financial product due to the result of compounding over a given time period

Term 3, Topic 4, Lesson 1

REVISION

Suggested lesson duration: 1,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners will have revised: • simple interest and hire purchase • compound interest and inflation • exchange rates.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. If there isn't a revision exercise in the textbook that you use, either use the revision exercise at the end of a Grade 10 textbook or items from a Grade 10 test on Finance.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	269			Qu's	252	11.1	369	9.1	376

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. This lesson takes the form of working through questions from two Grade 10 final exams. As you work through the solutions, remind learners or re-teach the concepts covered in previous years.
2. Although Finance is not a major part of the final examination, it will be useful to all learners after school. Everyone needs some knowledge of financial matters in their own lives – particularly once they start working.

DIRECT INSTRUCTION

1. Start the lesson by asking learners what they remember learning in Financial Mathematics in Grade 10. As topics are given, write them on the board. Tell learners to write the list in their exercise books. Ensure the following items are on the list:
   - Simple interest
   - Compound interest
   - Exchange rates
   - Hire purchase
   - Inflation.

   Ask learners questions to assess what they remember from last year. Tell them to take notes when you sum up (or add to) what they can tell you.

2. Ask: *What is the formula for simple interest?*
   ($\mathrm{A} = \mathrm{P}(1 + n.i)$)
   What do the A, P, n and i represent?
   (Accumulated or final amount;
   Principal or starting amount; number of years; interest rate).
3. Ask: *What is the formula for compound interest?*
   ($\mathrm{A} = \mathrm{P}(1 + i)^n$)
   What do the A, P, n and i represent?
   (Accumulated or final amount;
   Principal or starting amount; number of years; interest rate).

Although learners are likely to tell you that n represents number of years mention that this will not always be the case. It is best to think of n as the number of time periods that interest will be calculated. Tell learners that in Grade 11 they learn about interest that could be compounded monthly or weekly.

4. Ask: *What does it mean to buy an item on hire purchase?*
   (Pay a deposit, pay monthly instalments over a certain period of time usually at a high interest rate).
   Ask: *What type of interest is used for hire purchase?*
   (Simple)
5. Ask: *What does the term inflation mean?*
   (A general increase in prices and fall in the purchasing value of money).
   Ask: *What type of interest is used for calculating the rate of inflation?*
   (Compound)
   Why?
   (Because the new price each year needs to be taken into account; therefore it is interest on interest).

Learners should now have a summary of the financial mathematics they should already know.

6. Tell learners that you are going to spend the rest of the lesson doing some questions from previous Grade 10 examinations. Encourage learners to assist you in answering the questions and/or asking questions if they do not understand something.

Example 1	**Teaching notes**
Thando has R4 500 in his savings account. The bank pays him a compound interest rate of 4,25% p.a. Calculate the amount that Thando receives if he decides to withdraw the money after 30 months. 2012 Exemplar	This may seem like a straightforward question, but learners need to notice that Thando withdraws the money after 30 months which is not a complete number of full years. This needs to be adjusted accordingly to find the value of n Ask: *How many years is 30 months?* (2,5)
Solution: $A = P(1+i)^n$ $A = 4500(1+0,0425)^{2,5}$ $A = 4993,47$ Thando would have R4993,47	

Example 2	**Teaching notes**
The following advertisement appeared with regards to buying a bicycle on a hire purchase agreement loan: Purchase price: R5999 Required deposit: R600 Loan term: ONLY 18 months at 8% p.a. simple interest a) Calculate the monthly amount that a person must budget for, in order to pay for the bicycle.	Ask: *What needs to be the first calculation?* (Taking the deposit off the total amount) Remind learners that the deposit is paid upfront and is therefore not considered when adding on the interest. Ask: *What needs to be done once the deposit is subtracted? (*The total amount owing must be calculated using the simple interest formula) Ask: *What will n be?* (1,5) Ask: *What needs to be done with the new amount to find the monthly instalments?* (Divide the total by number of months).
b) How much interest does one have to pay over the full term of the loan? 2012 Exemplar	Remind learners of the difference between interest and interest rate. The interest is a total amount of money. Ask: *how would we calculate the interest paid?* (subtract the total 'borrowed' from the total paid).

Solution:

a) Loan amount required: R5999 − R600 = R5399

Total owing: $A = P(1 + n.i)$

$A = 5399(1 + (1{,}5)(0{,}08))$

$A = 6046{,}88$

Monthly payment: $\frac{6046{,}88}{18} = 335{,}94$

R335,94 will need to be budgeted

b) R6046,88 − R5399 = R647,88

Example	Teaching notes
The following information is given: 1 ounce = 28,35g $1 = R8,79 Calculate the rand value of a 1kg gold bar, if 1 ounce of gold is worth $978,34. 2012 Exemplar	Ask: *What do we need to find out first?* (How many ounces of gold are in a kg) Ask: *What calculation needs to be done next?* (Find what that will cost in dollars before changing it into the rands value.)

Solution:

1kg = 1000g

$\therefore \frac{1000}{28,35} = 35,2733....$

Cost in dollars = $= 35,2733.... \times 978,34 = 34509,347....$

Cost in rands = $34509,347.... \times 8,79 = 303337,16$

Remind learners to keep the answer in their calculator to ensure as accurate answer as possible.

This would be an ideal opportunity to explain why the strength of one's own currency is so important to the economy. Ask learners to calculate the cost of the same gold if the exchange rate was R12,43 to the dollar (R428 951,19). Quite a difference!

Example	**Teaching notes**
Zach likes to travel. He has saved R5000 as spending money for his vacation in Australia at the end of this year (2015). a) Zach looks up the exchange rate on the internet. Using the information in the table below, calculate how many Australian dollars Zach can buy for R5000.	The changing of one rate to another with the rate given should be straightforward for learners.
Zach plans another trip to Australia at the end of 2018 b) Assume that the average annual rate of inflation in South Africa will be 6,1% over the next three years. In 2018, what amount of money will be equivalent to the value of R5000 now?	Ask: *What calculation is required?* (Use 5000 as the principal amount and use compound interest to find what it will have increased to in 3 years' time).

Currency	**Equivalent value of R1**	**Rand equivalent of unit of currency**
Australian dollar	0,105058	9,518569

c) Zach plans to invest equal amounts into a savings account on 1 December 2016 and 1 December 2017 to have accumulated an amount of R5980 by 1 December 2018. If this account earns interest at 9% p.a. compounded annually, how much money should Zach deposit into the account on each occasion? NSC NOV 2015	Ask: *What is the unknown in this question?* (The principal amount that Zach needs to save). Ask: *Do we know the other values – A, n and i?* (Yes, but the *n* will change as the first amount will grow for two years and the second amount will grow for one year). Point out that the two amounts will have to be seen as separate savings. Tell learners that algebraic skills will be used in this calculation.

Solution:

a) R1 = 0,105058 AUS dollars

$\therefore R5000 \times 0,105058 = 525,29$

Zach will be able to buy \$525,29

b) $A = P(1+i)^n$

$A = 5000(1+0,061)^3$

$A = 5971,95$

c)
$$P(1+i)^n + P(1+i)^n = 5980$$
$$P(1+0,09)^2 + P(1+0,09)^1 = 5980$$
$$P(1,09)^2 + P(1,09)^1 = 5980$$
$$P((1,09)^2 + 1,09) = 5980$$
$$P = \frac{5980}{((1,09)^2 + 1,09)}$$
$$P = 2624,99$$

Ask how the first two terms can be simplified
(Take out a common factor P)

Zach will have to invest R2624,99 each year.

7. Ask directed questions so that you can ascertain learners' level of understanding.
   Ask learners if they have any questions.

8. Give learners an exercise to complete on their own.

9. Walk around the classroom as learners do the exercise. Support learners where necessary.

Term 3, Topic 4, Lesson 2

DEPRECIATION

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● differentiate between straight line and reducing balance depreciation ● solve problems involving depreciation.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson draw the graphs from point 4.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
2	275	1	240	4 5 6	266 269 271	11.9	392	9.2 9.3	381 385

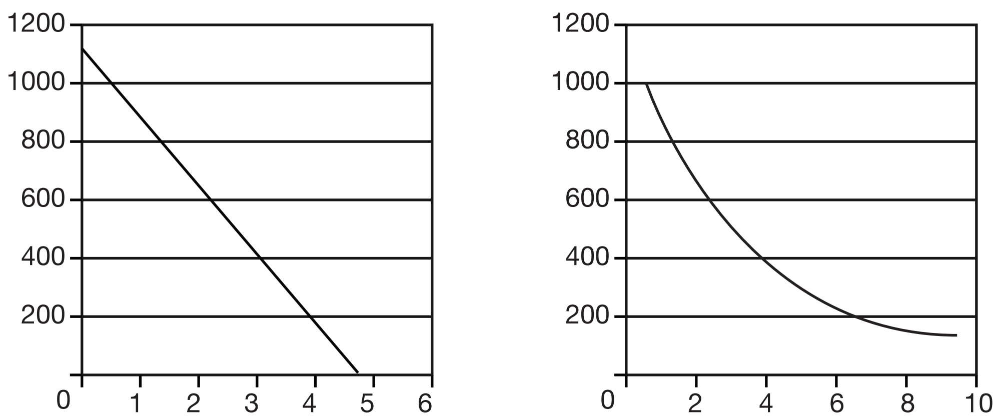
C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. All homes have items that will depreciate over time. It is important that learners understand this concept.

DIRECT INSTRUCTION

1. Start the lesson by asking learners if anyone knows what depreciation means and how it could affect us. Listen to a few responses before confirming a definition that learners can write in their exercise books.
2. Depreciation is a reduction in the value of an asset over time, particularly due to wear and tear. The current value of an asset after depreciation is often known as the book value.
3. Tell learners that there are two types of depreciation – one where the asset is eventually worth nothing at all and another where the asset's value decreases but it will always be worth something.
4. Show learners the two visual representations of depreciation:



5. Ask: *Can you see that the first type of depreciation will reach zero after a period of time?*
   Ask: *Can you see that the second type of depreciation is not going to get to zero?*

Ask:	*What is the second graph called and why it won't reach zero?* (It is an exponential function and zero represents the asymptote).

6. The first type of depreciation is called 'Straight Line' depreciation and the second type is called 'Reducing Balance' depreciation. Write each heading below its corresponding graph and ask learners to draw a sketch of the graph with its heading.

7. Refer to the two graphs and ask learners to note the following:
   - If an item depreciates on a straight-line balance it depreciates by the same amount each year.
   - If an item depreciates on a reducing balance it depreciates by a smaller amount with each year that goes by.

8. Ask: *Why do you think an item depreciates by a smaller amount each year?*
   (Each year it is worth less and each year it reduces by a percentage of its value at the beginning of that year).
   Ask: *Why can this value never (theoretically) be zero?*
   (A percentage of an amount can never be zero).

Point out why you used the word theoretically – the item will eventually be worth such a small amount that it will round to zero.

9. Tell learners that now you are going to discuss the formulae for each type with them. Remind them that simple interest produced a straight-line graph and that compound interest produced an exponential graph. It follows then that the formulae are linked to the formulae that have been used before.

10. Ask: *Which part of each formula do you think will be different for calculating depreciation*? (The addition will have to be subtraction to ensure the principal amount is less than the accumulated amount).

11. Ask learners to write down the two formulae:

Straight line depreciation	$A = P(1 - n.i)$
Reducing balance depreciation	$A = P(1 - i)^n$

12. Do the following worked example with learners. Learners should write it in their books.

Example 1	**Teaching notes**
A fridge costs R9999. Calculate what it will be worth in 5 years' time if it depreciates: a) on a reducing balance at 8% p.a. b) on a straight-line basis at 10% p.a.	Tell learners that this is questioning depreciation in a straightforward way and all that is required is substitution into the correct formula.
Solution: a) $A = P(1 - i)^n$ $A = 9999(1 - 0{,}08)^5$ $A = 6590{,}16$	b) $A = P(1 - n.i)$ $A = 9999(1 - (5)(0{,}1)$ $A = 4999{,}50$

13. Tell learners that you will now look at a few more ways that a question could be asked to develop their understanding of depreciation further.

Example 2	**Teaching notes**
The price of a new school bus is R540 000. The value of the bus decreases at 11% per annum according to the diminishing balance method. Calculate the value of the bus after 8 years. EC 2015	Tell learners that this is questioning depreciation in a straightforward way and all that is required is substitution into the correct formula.
Solution: $A = P(1-i)^n$ $A = 540000(1-0,11)^8$ $A = 212575,80$ The value of the bus is R212 575,80	
Example 3	**Teaching notes**
A cellphone is currently worth R3 037,50. It depreciated for 3 years at a rate of 25% per annum on a reducing balance method. adapted from EC 2016	Ask: *What is required?* (The original value – the 'P' in the formula) This is an important issue – learners tend to want to put the value known into P's place. Remind learners that in a depreciation question, the principal amount should always be larger than the final amount. This will require substitution and solving.
Solution: $A = P(1-i)^n$ $3037,50 = P(1-0,25)^3$ $\frac{3037,50}{(1-0,25)^3} = P$ $\therefore P = 7200$ The cellphone cost R7 200.	

Example 4	**Teaching notes**
A tractor bought for R120 000 depreciates to R11 090,41 after 12 years by using the reducing balance method. Calculate the rate of depreciation per annum. (The rate was fixed over the 12 years). NSC NOV 2014	Ask: *What is required?* (The rate of depreciation) This requires substitution and solving. Remind learners that when i has been found, it is the decimal version of the percentage. i will still have to be changed into a percentage.
Solution: $A = P(1-i)^n$ $11090,41 = 120000(1-i)^{12}$ $\frac{11090,41}{120000} = (1-i)^{12}$ $\sqrt[12]{\frac{11090,41}{120000}} = 1-i$ $\sqrt[12]{\frac{11090,41}{120000}} - 1 = -i$ $-0,17999\ldots = -i$ $\therefore i = 0,17999\ldots$ $\therefore$ the rate of depreciation is 18%.	

14. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
15. Give learners an exercise to complete on their own.
16. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.profitbooks.net/what-is-depreciation/

Term 3, Topic 4, Lesson 3

DIFFERENT PERIODS OF COMPOUND GROWTH AND DECAY (INCLUDING TIMELINES)

Suggested lesson duration: 3 hours

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners should be able to: ● solve complex problems with changing rates and withdrawals using timelines.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson write the table from point 4.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
3	279	2	249	1	255	11.3	374	9.5	393
5	285			2	258	11.4	377		
6	289					11.5	380		
						11.6	382		

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. Although there is only one main concept covered in this lesson, there are many new ideas.
2. The lesson has been designed to go through each idea in detail to assist learners to encounter each new idea on its own before combining them at the end.
3. Some textbooks only have one exercise and others have more. The textbooks with more exercises have assessed each of the new ideas as they have been taught. Check carefully when to stop teaching and when to allow learners to practice what has been taught. The more often learners work independently to do a few questions themselves, the better.

DIRECT INSTRUCTION

1. Start the lesson by telling learners that up until now they have only dealt with interest rates on an annual basis. In fact it is possible to compound interest on a more frequent basis, for example, semi-annually (twice a year), monthly (12 times a year), weekly (52 times per year) or even daily (365 times per year).
2. Explain that this means that the n does not always represent a year (as mentioned in the first lesson in this section). The n represents the number of times that the interest will be calculated.

   For example: If interest is offered monthly and money is being saved for 3 years, what would n represent? (36 – there are 36 months in 3 years; 3 years $\times$ 12)

3. The interest rate is always given per annum, but now we are saying that the rate is actually monthly. This means that if the interest rate is per annum, but it is offered monthly then it needs to be divided by 12 because it will be calculated every month.

4. Tell learners to record the summary of what needs to be done when interest is compounded over different time periods:

	n	i
annually	Number of years	i
semi-annually	Number of years $\times 2$	$\frac{i}{2}$
quarterly	Number of years $\times 4$	$\frac{i}{4}$
monthly	Number of years $\times 12$	$\frac{i}{12}$
weekly	Number of years $\times 52$	$\frac{i}{52}$
daily	Number of years $\times 365$	$\frac{i}{356}$

5. Ask learners to calculate the following possibilities:
Find the future value of an investment of R5 000 after 4 years if the interest rate of 7,5% is compounded:

a) annually b) semi-annually c) quarterly

d) monthly e) weekly f) daily

Solutions:

a) annually: $A = 5000(1+0,075)^4$ $A = R6677,35$	b) semi-annually: $A = 5000\left(1+\frac{0,075}{2}\right)^8$ $A = R6712,35$	c) quarterly: $A = 5000\left(1+\frac{0,075}{4}\right)^{16}$ $A = R6730,57$
d) monthly: $A = 5000\left(1+\frac{0,075}{12}\right)^{48}$ $A = R6743,00$	e) weekly: $A = 5000\left(1+\frac{0,075}{52}\right)^{208}$ $A = R6747,84$	f) daily: $A = 5000\left(1+\frac{0,075}{356}\right)^{1424}$ $A = R6749,08$

6. Ask: *What do you notice about each of these amounts and their periods of interest earned?* (The more often that interest is calculated and added, the more money is saved).

7. Learners need to read carefully when answering these types of questions and to be aware of the period of saving time.

   Ask: *If a sum of money was saved for 5,5 years at an interest compounded monthly, what would 'n' be equal to?* (5,5 × 12 = 66)

8. Do the following two worked examples with learners. Learners should write the worked examples in their exercise books.

Example 1	Teaching notes
What amount of money should Godfrey invest for 6 years at an interest rate of 8,8% per annum compounded semi-annually to have an amount of R15 000 saved?	Ask: *What value in the formula is required?* (P – the principal amount) Ask: *What is 'n' equal to - the number of interest calculations?* (6 × 2 = 12) Ask: *What is 'i' equal to – the amount of interest given within each time period?* $\left(\frac{0,088}{2}\right)$

Solution:

$$15000 = P\left(1+\frac{0,088}{2}\right)^{12}$$

$$\frac{15000}{\left(1+\frac{0,088}{2}\right)^{12}} = P$$

$$\therefore P = R8947,16$$

Example 2	Teaching notes
Cyril has R5000 and wants to invest it so that it grows to R8000. He has 4 years to do so. What interest rate, compounded monthly, needs to be offered for this to be possible?	Ask: *What value in the formula is required?* (i – the interest rate) Ask: *What must we remember to do with the interest rate?* (Divide by 12 as it is compounded monthly). Ask: *What is 'n' equal to - the number of interest calculations?* (4 x 12 = 48)

Solution:

$$8000 = 5000\left(1+\frac{i}{12}\right)^{48}$$

$$\frac{8}{5} = \left(1+\frac{i}{12}\right)^{48}$$

$$\sqrt[48]{\frac{8}{5}} - 1 = \frac{i}{12}$$

$$\left(\sqrt[48]{\frac{8}{5}} - 1\right) \times 12 = i$$

$$0,11807\ldots = i$$

$$\therefore 11,81\%$$

9. When working in reverse (finding the principal amount as in the first example) learners should note the following:

   Learners should write down the compound interest formula then make P the subject of the formula:

$$A = P(1+i)^{n}$$
$$\frac{A}{(1+i)^{n}} = P$$

   At this step, ask: *What can you do to have no fractions in the formula?*

   (Use the laws of exponents to have the power in the numerator position).

$$A(1+i)^{-n} = P$$

   Learners can use this version of the formula if they wish to when finding the principal amount. It will also be useful later in the lesson.

10. Calculations regarding saving money and paying back a loan are not always as simple as in previous grades. In the real world, someone may put some money in savings then need to take a portion of it out. The bank may change the interest rate offered during the course of some money being saved.

11. These issues could make a calculation more complicated. To assist us in solving these problems, we are going to use timelines. Timelines are visual representations of all the changes that happen during the entire period of time that an amount of money is saved, or a loan is paid back.

12. First we will look at only the interest rate being changed during the savings period.

13. Use the following example to illustrate the use of a timeline.

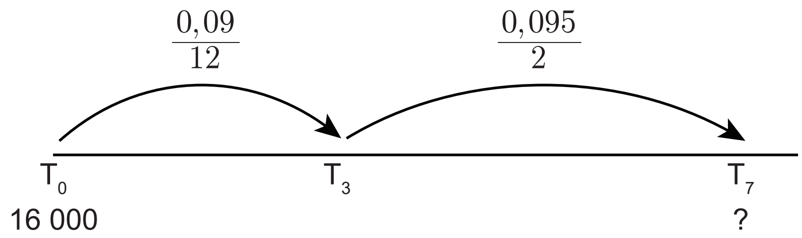
    This example is used as the introduction to use timelines. Any examples that follow will not be done in such detail. Ensure you are comfortable with the process so that each question requiring a timeline can be done in a similar way from now on.

Learners should take the worked example down, making notes as they do so.

Example 3	**Teaching notes**
Thembi invests R16 000 for a period of 7 years into an account that pays 9% p.a. compounded monthly for the first 3 years. The interest rate then changes to 9,5% p.a. compounded semi-annually. Calculate the future value of her investment.	Learners should highlight or underline all the key figures (amounts, interest rates and years). The different times mentioned over the entire period will be written as follows: T_1 or T_5 for example, which mean Term 1 (year 1) or Term 5 (year 5). The beginning of the entire period is represented as T_0. Learners should see this as the very first day, when Thembi goes into the bank with her money and a year has not yet passed (zero years).

Work through the making of the timeline as follows:

1. List all the key times (T_0, T_3, T_7)
2. Fill in the invested amount at T_0 (R16 000)
3. Consider the interest rates. Use a curve to show all the years that share the same interest rate and fill in the rate above each curve (including the amount it will be divided by according to the compounding period).



Learners should tick off each piece of information as they put it onto the timeline to ensure nothing is left out.
Learners should proceed in the same order each time they make a timeline as this will assist them in following a certain routine and therefore also not leaving something out.
Tell learners to notice the direction of the arrows – they are moving forward – the money is growing.

Once the time line has been drawn up discuss with learners:
Can you see that we will need to know how much money there is at the end of the 3 years to know the new principal amount which will help us to calculate what is available at the end of the 7 years?
Ask for a volunteer to: *Write the formula for the first 3 years on the board and fill in the appropriate values.*

$$A = P(1+i)^n$$
$$A = 16000\left(1+\frac{0,09}{12}\right)^{36}$$

Discuss each of the values, ensuring learners know where they all come from.
Ask: *Do you all agree that this amount, once calculated, will be how much money is in Thembi's account before the interest rate changes for the following 4 years?*
In other words, it will be the new principal amount.
Ask for a volunteer to: *Write the formula for the next 4 years on the board and fill in the appropriate values? For now we will call the unknown amount P.*

$$A = P(1+i)^n$$
$$A = P\left(1+\frac{0,095}{2}\right)^{8}$$

Discuss each of the values, ensuring learners know where they all come from.
Tell learners that instead of doing TWO calculations you are going to show them how to do it into one calculation:
Rewrite the last calculation written, but leave a bigger space when writing the P.

$$A = P\left(1+\frac{0,095}{2}\right)^{8}$$

Remind learners that they agreed that the P would be the first calculation.
Erase the P and replace it with the first calculation:

$$A = 16000\left(1+\frac{0,09}{12}\right)^{36}\left(1+\frac{0,095}{2}\right)^{8}$$

Learners may find it easier to see it written as follows:

$$A = \left[16000\left(1+\frac{0,09}{12}\right)^{36}\right]\left(1+\frac{0,095}{2}\right)^{8}$$

Remind learners that the square brackets represent the 'new' P. The square brackets are not essential when using the calculator.
This can be done as it stands on the calculator. Ask learners to do the calculation.

$$A = 30351,08$$

Thembi will have R30 351,08 at the end of the 7 years.

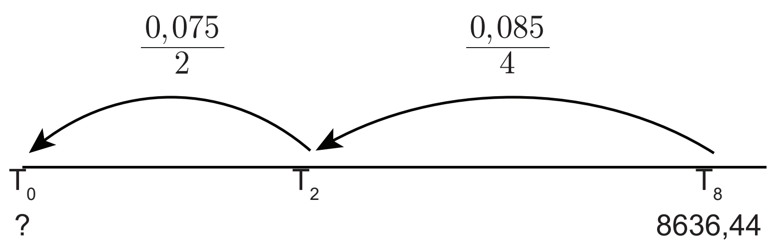
14. Tell learners that a similar question could be asked when working in reverse is required (finding the amount that was initially saved).
    Remind learners of the formula derived when finding the value of P is required:

$$A(1+i)^{-n}=P$$

15. Do this worked example with learners:

Example 4	Teaching notes
Xolile invested a certain sum of money for 8 years at 7,5% p.a. compounded semi-annually for the first 2 years and 8,5% p.a. compounded quarterly for the next 6 years. At the end of 8 years her money had grown to R8636,44. Find the amount that Xolile invested.	Remind learners to highlight or underline the key figures. Ask: *What are the time periods?* (T_0, T_2, T_8) Ask: *What are the different interest rates?* $\left(\frac{0,075}{2};\frac{0,085}{4}\right)$ Ask: *Where will the amount of R8636,44 be placed on the timeline?* (At the end because it is the accumulated amount. Once the timeline has been drawn, show learners that we will be starting at the back and working in reverse. Due to P being the subject of the formula, remember that the exponents will be negative.

Solution:



Tell learners to notice the direction of the arrows – they are going backwards – the money is decreasing.

$$P=8636,44\left(1+\frac{0,085}{4}\right)^{-24}\left(1+\frac{0,075}{2}\right)^{-4}$$
$$P=4500$$

Xolile started with R4 500

16. Ask if anyone has any questions or needs anything to be explained again.

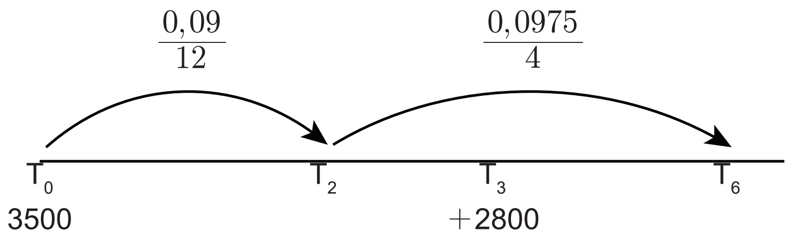
17. Tell learners that not only interest rates could change, but the people saving could decide to make an additional payment during the savings period or they could need the money and need to make a withdrawal. This will be dealt with now.

18. Tell learners that timelines become more important as the situation gets more complicated.

19. Do the following worked example with learners. While drawing the timeline, ensure all learners understand where each number placed on the timeline comes from and why it has been placed in a certain position. When using the timeline to do the calculation, ensure that all learners understand each part of the calculation.
    Learners should write the worked example in their books.

Example 5	**Teaching notes**
Godfrey deposits R3 500 into a savings account. Three years later he deposits a further R2 800. The interest rate for the first two years is 9% p.a. compounded monthly. The interest rate changes to 9,75% p.a. compounded quarterly for the final 4 years of the investment period. How much does Godfrey have at the end of the 6th year?	Remind learners to highlight or underline the key figures. Ask: W*hat are the time periods?* (T_0, T_2, T_3, T_6) Tell learners to note that the time periods are not only linked to interest rate changes but also to a further payment. It is important that any year where something happens needs to be noted. Ask: *What are the different interest rates?* $\left(\frac{0,09}{12}; \frac{0,0975}{4}\right)$

Solution:



> Important advice for learners:
> When any additional payments or withdrawals are made, it is easier to deal with each amount ONE AT A TIME all the way to the end of the investment period and then add (or subtract) the new amount along with the interest involved.
> In this case: First work with what the R3 500 would have grown to in the 6 years as if that was all that was saved. This will then be added to the second amount involved (R2 800) along with any interest earned.

$$A = 3500\left(1 + \frac{0,09}{12}\right)^{24}\left(1 + \frac{0,0975}{4}\right)^{16} + 2800\left(1 + \frac{0,0975}{4}\right)^{12}$$

> Point out that the R2 800 was only in the account for three years – hence the n value is 4 × 3.

$$A = 9894,15$$

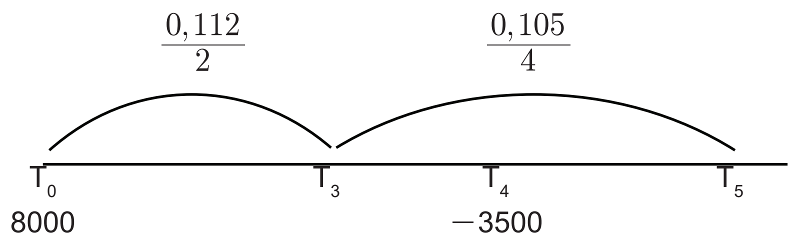
Godfrey will have R9894,15 at the end of the investment period.

20. Ask learners if they have any questions or want anything explained again.

21. Tell learners that you will now do a worked example which includes a withdrawal during the period of the investment.

Example 6	**Teaching notes**
Thomas invests R8 000. After 4 years he needs R3 500 for an emergency so withdraws from the account. The interest rate is 11,2% compounded semi-annually for 3 years then changes to 10,5% compounded quarterly for the remaining 2 years of the investment. How much money will be available at the end of the investment period?	Remind learners to highlight or underline the key figures. Ask: *What are the time periods?* (T_0, T_3, T_4, T_5) Ask: *What are the different interest rates?* $\left(\frac{0,112}{2}; \frac{0,105}{4}\right)$ Remind learners that each amount will be dealt with separately. Ask: *What makes this example different from the previous example?* (The money is being taken out of the account) Ask: W*ill Thomas just lose the R3 500?* (No – he will also lose the interest that the money would have gained so this needs to be added onto the R3 500).

Solution:



$$A = 8000\left(1+\frac{0,112}{2}\right)^{6}\left(1+\frac{0,105}{4}\right)^{8} - 3500\left(1+\frac{0,105}{4}\right)^{4}$$
$$A = 9766,71$$

There will be R9766,71 available at the end of the investment period.

22. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

23. Give learners an exercise to complete on their own.

24. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.siyavula.com/read/maths/grade-11/finance-growth-and-decay/09-finance-growth-and-decay-03
(This is from the Siyavula book for schools not already using the Siyavula book)

Term 3, Topic 4, Lesson 4

NOMINAL AND EFFECTIVE INTEREST RATES

Suggested lesson duration: 1,5 hours

POLICY AND OUTCOMES

A

CAPS Page Number	37

Lesson Objectives

By the end of the lesson, learners should be able to:

- explain the difference between nominal and effective interest rates
- convert between nominal and effective rates
- solve problems involving nominal and effective interest rates.

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson write the two questions from point 1.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	282			3	262	11.8	388	9.6	397

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. A better understanding of how interest rates work will enhance learners' ability to manage their own finances (if not now, in the future when they begin to earn money).
2. Where possible, relate examples to everyday life and associate concepts to what learners may know.

DIRECT INSTRUCTION

1. Start the lesson by asking learners to do these two calculations:

   a) R6 700 is saved for a year at an interest rate of 7% p.a. compounded annually. Find the future value.

   b) R6 700 is saved for a year at an interest rate of 7% p.a. compounded monthly. Find the future value.

   a) R7 169,00 b) R7 184,34

   Tell learners that the interest rate of 7% mentioned in these examples is the nominal rate. The nominal rate is the interest rate quoted by the financial institutions such as banks. Even when banks quote 7% p.a. compounded monthly, the amount of interest really earned is a little more than that.

3. Continue investigating the above examples. Ask learners to find the difference in money made with the two options (R7 184,34 − R7 169,00 = R15,34).

4. Find the percentage increase:

$$\frac{15,34}{6700}\times 100 = 0,22895\ldots\%$$

   Tell learners that by compounding the interest monthly, we have effectively earned 0,229% more than the 7% quoted. In other words, the effective interest rate is 7,229%.

5. Learners should write down the definitions of nominal and effective interest:
   Nominal interest rate: The interest rate quoted which does not consider the compounding period.
   Effective interest rate: The rate of interest effectively earned as it takes the compounding period into account.
   The more often compounding occurs, the higher the effective interest rate.

6. There is a formula that can change one type of rate of interest into the other type.

7. Write the formula on the board:

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$$

8. Learners should record the formula and highlight it for easy reference. Tell learners that the formula will not be given to them in assessments and that they need to know it.

9. Important issue regarding nominal and effective interest calculations:
ONLY one year is considered even if an investment of more than that is mentioned.
The formula will give an effective rate over the course of one year.

10. Summarise what each variable represents. Learners should write it down.

i_{eff} – effective interest rate i_{nom} – nominal interest rate n – number of compounding periods in ONE year

11. Do the following three worked examples from past examinations. Learners should write them in their exercise books:

Example 1	Teaching notes
Determine the effective interest rate if an investment earns interest at a nominal interest rate of 11,5% per annum, compounded quarterly. EC 2015	Remind learners that their calculation must represent one year. It is important to note the compounding period. Ask: *What is the compounding period?* (Quarterly) Ask: *How many quarters in one year?* (4) (This will be the n value.)
Solution: $i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$ $i_{eff} = \left(1 + \frac{0,115}{4}\right)^4 - 1$ $i_{eff} = 0,120055\ldots.$ $\therefore i_{eff} = 12,01\%$	Remind learners to always check that their effective rate is a little higher than the nominal rate.

Example	Teaching notes
Calculate the effective interest rate if interest is 9,8% p.a. compounded monthly. NSC NOV 2014	Remind learners that their calculation must represent one year. It is important to note the compounding period. Ask: *What is the compounding period?* (Monthly) Ask: *how many months in one year?* (12) (This will be the n value.)
Solution: $i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$ $i_{eff} = \left(1 + \frac{0,098}{12}\right)^{12} - 1$ $i_{eff} = 0,102523\ldots$ $\therefore i_{eff} = 10,25\%$	

Example	Teaching notes
John invested R120 000. He is quoted a nominal interest rate of 7,2% p.a. compounded monthly. a) Calculate the effective interest rate p.a. correct to three decimal places.	Ask: *what is the compounding period?* (monthly) Ask: *how many months in one year?* (12) This will be the n value.
b) Use the effective interest rate to calculate the value of John's investment if he invested the money for three years.	Remind learners that the effective interest rate is per year. They will therefore not divide it by 12 now – it is already annually due to the conversion.
c) Suppose John invests his money for a total of 4 years, but after 18 months he makes a withdrawal of R20 000. How much will he receive at the end of 4 years? EC 2016	A timeline will work best for this. Ask: *what are the time periods?* ($T_0, T_{1,5}, T_4$) Remind learners that each amount will be dealt with separately.

Solution:

a)

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$$

$$i_{eff} = \left(1 + \frac{0,072}{12}\right)^{12} - 1$$

$$i_{eff} = 0,074424....$$

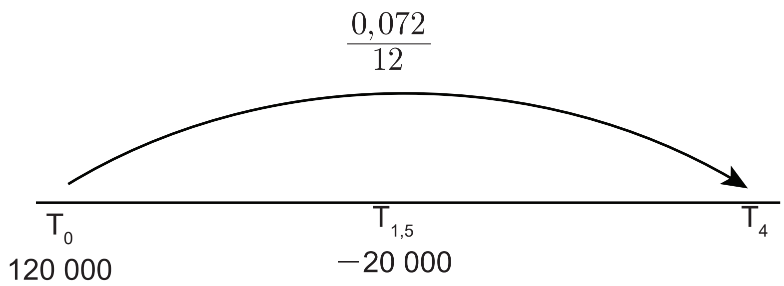
$$\therefore i_{eff} = 7,442\%$$

b)

$$A = 120000(1 + 0,07442)^3$$

$$A = 148834,46$$

c)



$$A = 120000\left(1 + \frac{0,072}{12}\right)^{48} - 20000\left(1 + \frac{0,072}{12}\right)^{30}$$

$$A = 135981,73$$

12. Ask directed so that you can ascertain learners' level of understanding.
    Ask learners if they have any questions.

13. Give learners an exercise to complete on their own.

14. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=-ikxtJELjLY

Term 3, Topic 4, Lesson 5

REVISION AND CONSOLIDATION

Suggested lesson duration: 1 hour

A POLICY AND OUTCOMES

CAPS Page Number	37
Lesson Objectives By the end of the lesson, learners will have revised: • all the concepts covered in the previous lessons.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plan and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev S Ch	291 292	Rev	250	Qu's Ch 10	272 294	11.2 Rev	371 394	9.4 9.7	388 399

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. Ask learners to recap what they have learned in this section. Spend time pointing out issues that you know are important as well as problems that you encountered from your own learners.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

This lesson comprises three fully worked examples from past papers covering all the concepts in this topic. As you work through these with the learners, it is important to frequently talk about as many concepts as possible, and to use the correct terminology repeatedly. For example, use the words simple interest, compound interest, inflation, depreciation, nominal and effective interest rates wherever possible, constantly reminding learners what they have already learnt.

Say: *I am going to do two worked examples from past exam papers with you. You should write them down as I do them, taking notes at the same time.*

Example: 1 NSC NOV 2016	**Teaching notes:**
1. A machine costs R25 000 in 2016. Calculate the book value of the machine after 4 years if it depreciates at 9%p.a. according to the reducing balance method.	Ask: W*hat formula will we use?* (The reducing balance depreciation formula $A=P(1-i)^n$) Ask: W*hat information has been given?* (P – principal amount, i – interest rate and n – number of years).
Solution: $A=P(1-i)^n$ $A=25000(1-0,09)^4$ $A=17143,74$	

2. The nominal interest rate of an investment is 12,35%p.a. compounded monthly. Calculate the effective interest rate.	Remind learners that their calculation must represent one year. It is important to note the compounding period. Ask: *What is the compounding period?* (monthly) Ask: *How many months in one year?* (12) This will be the n value.

Solution:

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$$
$$i_{eff} = \left(1 + \frac{0,1235}{12}\right)^{12} - 1$$
$$i_{eff} = 0,130736\ldots.$$
$$\therefore i_{eff} = 13,07\%$$

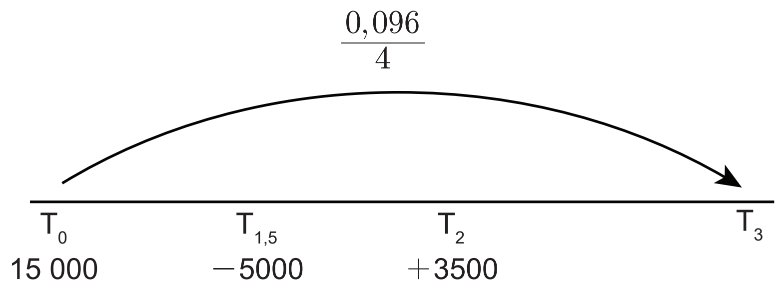
3. The value of a property increased from R145 000 to R221 292,32 over 6 years. Calculate the average annual rate of increase of the property over 6 years.	Ask: *What formula will be used?* (The compound interest as this situation is about inflation) Ask: *What information has been given?* (P – principal amount, A – final amount and n – number of years) Remind learners to convert the decimal they get for i back to a percentage.

Solution:

$$A = P(1+i)^n$$
$$221292,32 = 145000(1+i)^6$$
$$\frac{221292,32}{145000} = (1+i)^6$$
$$\sqrt[6]{\frac{221292,32}{145000}} = 1+i$$
$$\sqrt[6]{\frac{221292,32}{145000}} - 1 = i$$
$$\sqrt[6]{\frac{221292,32}{145000}} - 1 = i$$
$$0,07300000324\ldots = i$$
$$\therefore i = 7,3\%$$

4. Tebogo made an initial deposit of R15 000 into an account that paid interest at 9,6%p.a. compounded quarterly. Six months later she withdrew R5 000 from the account. Two years after the initial deposit she deposited another R3 500 into this account. How much does she have in the account 3 years after her initial deposit?	Remind learners to highlight or underline the key figures. Ask: *What are the time periods?* ($T_0, T_{0,5}, T_2, T_3$) Ask: *What amounts have been paid and withdrawn?* (15 000 at T_0, -5000 at $T_{0,5}$, $+3500$ at T_2) Ask learners to draw their own timeline before you do it with them on the board.

Solution:



$$A = 15000\left(1+\frac{0,096}{4}\right)^{12} - 5000\left(1+\frac{0,096}{4}\right)^{10} + 3500\left(1+\frac{0,096}{4}\right)^{4}$$
$$A = 17448,46$$

Example: NSC NOV 2015	**Teaching notes:**
1. A school buys tablets at a total cost of R140 000. If the average rate of inflation is 6,1% p.a. over the next 4 years, determine the cost of replacing these tablets in 4 years' time.	Ask: *What formula will be used?* (The compound interest as this situation is about inflation). Ask: W*hat information has been given?* (P – principal amount, i – the interest rate and n – number of years).

Solution:

$$A = P(1+i)^n$$
$$A = 140000(1+0,061)^4$$
$$A = 177414,69$$

2. An investment earns interest at a rate of 7% per annum compounded semi-annually. Calculate the effective annual interest rate on this investment.	Remind learners that their calculation must represent one year. It is important to note the compounding period. Ask: *What is the compounding period?* (Semi-annually) Ask: *How many half years in one year?* (2) This will be the n value.

Solution:

$$i_{eff} = \left(1 + \frac{i_{nom}}{n}\right)^n - 1$$

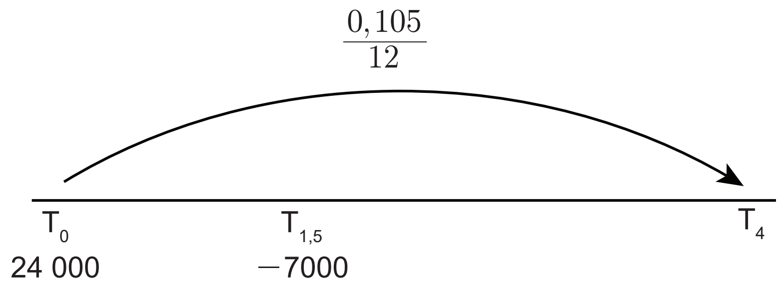
$$i_{eff} = \left(1 + \frac{0,07}{2}\right)^2 - 1$$

$$i_{eff} = 0,071225....$$

$$\therefore i_{eff} = 7,12\%$$

3. A savings account was opened with an initial deposit of R24 000. Eighteen months later, R7 000 was withdrawn from the account. Calculate how much money will be in the savings account at the end of 4 years if the interest rate was 10,5% p.a. compounded monthly.	Remind learners to highlight or underline the key figures. Ask: *What are the time periods?* ($T_0, T_{1,5}, T_4$) Ask: *What amounts have been paid and withdrawn?* (24 000 at T_0 , -7000 at $T_{1,5}$) Ask learners to draw their own timeline before you do it with them on the board.

Solution:



$$A = 24000\left(1 + \frac{0,105}{12}\right)^{48} - 5000\left(1 + \frac{0,096}{4}\right)^{10} + 3500\left(1 + \frac{0,096}{4}\right)^{4}$$

$$A = 27369,56$$

3. A car costing R198 000 has a book value of R102 755,34 after 3 years. If the value of the car depreciates at $r\%$ p.a. on a reducing balance, calculate r .	Ask: W*hat formula will we use?* (The reducing balance depreciation formula $A = P(1-i)^n$) Ask: W*hat information has been given?* (P – principal amount, A – the final amount after depreciation and n – number of years). Remind learners to convert the decimal they get for i back to a percentage.

Solution:

$$102755,34 = 198000(1-i)^3$$
$$\frac{102755,34}{198000} = (1-i)^3$$
$$\sqrt[3]{\frac{102755,34}{198000}} = 1-i$$
$$\sqrt[3]{\frac{102755,34}{198000}} - 1 = -i$$
$$-0,1963880\ldots = -i$$
$$\therefore i = 19,64\%$$

1. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
2. Give learners an exercise to complete on their own.
3. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=asMmXiFU_nc

TOPIC OVERVIEW

TOPIC 5: PROBABILITY

A TOPIC OVERVIEW

- This topic is the fifth of five topics in Term 3.
- This topic runs for two weeks (9 hours).
- It is presented over six lessons.
- The lessons have been divided according to sub-topics, not according to one school lesson. An approximate time has been allocated to each lesson (which will total nine hours). For example, one lesson in this topic could take two school lessons. Plan according to your school's timetable.
- Probability counts 13% of the final Paper 1 examination.
- Traditionally, probability either gets taught extremely well or almost not at all. If you feel that your knowledge is not as good as it could be, take the time to watch videos and read up on the concepts required.
- Watch the following 3-minute video for some inspiration: https://www.ted.com/talks/arthur_benjamin_s_formula_for_changing_math_education (Arthur Benjamin is a professor of mathematics in the United States. He discusses the fact that most topics done in school mathematics generally lead to being able to learn calculus. However, Benjamin believes that statistics and probability are in fact more important and that calculus can always be studied in more detail by students of mathematics who go on to study mathematics at tertiary level).

Breakdown of topic into 6 lessons:

	Lesson title	Suggested time (hours)		Lesson title	Suggested time (hours)
1	Revision	1,5	4	Contingency tables	1,5
2	Tree Diagrams	2	5	Dependent and Independent events	1
3	Venn Diagrams	2	6	Revision and Consolidation	1

SEQUENTIAL TABLE

B

GRADE 10 and Senior phase	GRADE 11	GRADE 12
LOOKING BACK	**CURRENT**	**LOOKING FORWARD**
• Compare relative frequency and theoretical probability • Use of Venn diagrams to solve problems • Mutually exclusive events • Complementary events • Derive and using the identity: $P(A\ or\ B) = P(A) + P(B) – P(A\ and\ B)$	• Revision of addition rule for mutually exclusive events • Dependent and independent events • Use of Venn diagrams, tree diagrams and contingency tables to solve problems.	• The fundamental counting principle • Probability problems using the fundamental counting principle.

WHAT THE NSC DIAGNOSTIC REPORTS TELL US

C

According to **NSC Diagnostic Reports** there are several issues pertaining to Probability.

These include:

- Confusion between n(A) and P(A)
- Not reading from a contingency table correctly
- No understanding of *mutually exclusive events* and *independent events*.
- Incorrect use of notation (for example, P(0,2)).

It is important that you, as the teacher, keep these issues in mind when teaching this section.

While teaching Probability, explain the concepts in depth and always use correct notation.

ASSESSMENT OF THE TOPIC

D

- CAPS formal assessment requirements for Term 3:
  - Two tests
- Two tests, with memoranda, are provided in the Resource Pack. The tests are aligned to CAPS in every respect, including the four cognitive levels as required by CAPS (page 53).
- The questions usually take the form of problems relating to the probability of an event and can include Venn diagrams, tree diagrams or contingency tables.
- Monitor each learner's progress to assess (informally) their grasp of the concepts. This information can form the basis of feedback to the learners and will provide you valuable information regarding support and interventions required.

E

MATHEMATICAL VOCABULARY

Be sure to teach the following vocabulary at the appropriate place in the topic:

Term	Explanation
probability	The likelihood or chance of something happening A probability answer is ALWAYS in the range: $0 \leq x \leq 1$
trial/experiment	The process of trying something out to find the chance (probability) of an event occurring For example: Tossing a coin 100 times
outcome	A possible result from an experiment For example: 'tails' is one of two possible outcomes when tossing a coin
sample space	The **sample space** of an experiment is the set of all possible outcomes of that experiment
experimental probability	The result of doing an experiment to ind the chances of an event occurring For example: An experiment was conducted to see how many tails appeared when a coin was tossed 100 times. The result was $\frac{47}{100}$
relative frequency	The outcome of an experiment In the above example $\frac{47}{100}$ is the relative frequency
theoretical probability	The probability of an event happening using knowledge of numbers. The theoretical calculation $P(A) = \frac{n(A)}{n(S)}$
tree diagram	Method used for counting the number of possible outcomes of an event The last column of the tree diagram shows all the possible outcomes
contingency table	Table showing the distribution of one variable in rows and another in columns, used to study the correlation between the two variables
Venn diagram	A representation of mathematical or logical sets of information In a Venn diagram, the position and overlapping of circles are used to indicate the relationships between different sets of information
union	The set of all outcomes that occur in at least one of the events Key word: or
intersection	The set of outcomes that occur in all the events Key word: and
mutually exclusive events	Events with no outcomes in common (no intersection)

complementary events	Mutually exclusive events that contain all the outcomes between them
independent events	Two events where the outcome of one event does not affect the outcome of the other
dependent events	The outcome of one event affects the outcome of the next event

Term 3, Topic 5, Lesson 1

REVISION OF PREVIOUS WORK

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	38

Lesson Objectives

By the end of the lesson, learners will have revised:

- theoretical probability
- Venn diagrams
- mutually exclusive and complementary events.

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 19 from the Resource Pack.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		**PLATINUM**		**VIA AFRIKA**		**CLEVER**		**EVERYTHING MATHS (SIYAVULA)**	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
1	248	1	255	Qu's	274	12.1	398	10.1	405

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. Most of this lesson takes the form of going through questions from two Grade 10 final exams. As you work the solutions are worked through remind or re-teach learners all the concepts covered in previous years.

DIRECT INSTRUCTION

1. Ask learners what they remember learning in probability in Grade 10. As concepts are given, write them on the board. Learners should write the list in their exercise books. Ensure that the following items are on the list.
   - Venn diagrams
   - Complementary events
   - Mutually exhaustive events
   - Dependent and Independent events
   - The identity, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

   > Learners may mention more concepts like tree diagrams and how to calculate probability or that a probability answer must always be a number from one to zero.
   > All of these suggestions are acceptable. The above list, however, relates directly to Grade 10 work and is what we will be revising in this lesson.
   > Ask learners questions to assess exactly what they remember from Grade 10.
   > Tell learners to take notes if they feel it will be helpful.

2. Ask: W*hat are complementary events?*
   (Events with nothing in common that contain all the outcomes between them).
3. Ask: *What are mutually exclusive events?*
   (Events with nothing in common).
4. Draw an example of a Venn diagram on the board and explain what it means.
5. Say: *Describe dependent and independent events with the use of an example.*
   (Independent events occur when the outcome of one event does not affect the other one happening, example: flipping a coin. Dependent events occur when the outcome of one event affects the other event.
   Example: drawing a card from a pack of cards and not putting it back, then drawing a second card).

6. Spend the rest of the lesson doing some questions from previous Grade 10 examinations. Encourage learners to assist you in answering the questions and/or asking questions if they do not understand something.
(These questions are taken from the 2015 and 2017 final examinations)

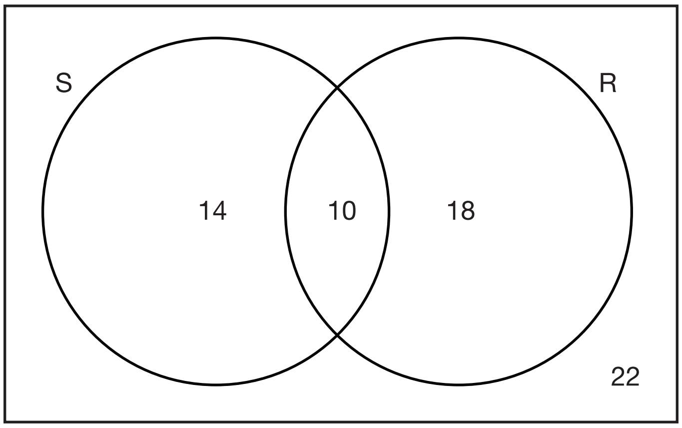
> An enlarged version of the table used in Example 2 is available in the Resource Pack. Resource 19.

Example 1	**Teaching notes**
At a school there are 64 boys in Grade 10. Their sport preferences are: • 24 boys play soccer • 28 boys play rugby • 10 boys play both rugby and soccer • 22 boys do not play soccer or rugby a) Represent the information above in a Venn diagram	Ask: *How many sets are represented?* (2 – soccer and rugby) Ask: *Are the sets inclusive?* (Yes – there are boys that like both soccer and rugby). Learners should draw the frame and the two overlapping circles. Ask: *Which part of the Venn diagram must always be completed first?* (The intersection) Tell learners to find the information that describes this (10 boys play both rugby and soccer) and to fill it in. Ask: *How many boys like ONLY soccer?* Ask: *How many boys like ONLY rugby?* (14 for soccer because $24 - 10 = 14$ and 18 for rugby because $28 - 10 = 18$). Fill this in then finally fill in those who play neither rugby nor soccer on the outside of the sets.

b) Calculate the probability that a Grade 10 boy at the school, selected at random, plays: (i) soccer and rugby (ii) soccer or rugby	Remind learners that the total amount of boys will be the sample space and therefore the denominator. Remind learners that 'or is more' – the probability of one or another event happening will always be better than one event and another event happening.
c) Are the events a Grade 10 boy plays soccer at the school and a Grade 10 boy plays rugby at the school, mutually exclusive? Justify your answer.	Ask: *Explain what mutually exclusive means?* (No intersection).

Solution:

a)



b) (i) $P(S\ and\ R) = \frac{10}{64} = \frac{5}{32}$

(ii) $P(S\ or\ R) = \frac{42}{64} = \frac{21}{32}$

c) The events are not mutually exclusive as they have something in common (an intersection).

$P(S\ and\ R) \neq 0$

Example 2

One morning Samuel conducted a survey in his residential area to establish how many passengers, excluding the driver, travel in a car. The results are shown in the table below:

Number of passengers, excluding the driver	0	1	2	3	4
Number of cars	7	11	6	5	1

Calculate the probability that, excluding the driver, there are more than two passengers in a car.

Teaching notes

Ask: *What do we need to calculate first if probability is required?*
(The total number of cars).
Ask: *What parts of the table should be our next focus?*
(The blocks representing more than two passengers, in other words the 3 and 4).

Solution:

P (more than 2 passengers) $= \frac{6}{30} = \frac{1}{5}$

(Total cars: 7 + 11 + 6 + 5 + 1 and total cars with 3 or 4 passengers: 5 + 1)

Example 3

If you throw two dice at the same time, the probability that a six will be shown on one of the dice is $\frac{10}{36}$ and the probability that a six will be shown on both the dice, is $\frac{1}{36}$. What is the probability that a six will NOT be shown on either of the dice when you throw two dice at the same time?

Teaching notes

Tell learners that the word 'NOT' is very important.
Ask: *What concept of probability does this question imply?* (complementary events)
Ask: W*hat are complementary events ?*
(Mutually exclusive events that contain all the outcomes between them. They are the only two possible outcomes).
Say: *It may be useful to use the probability of getting a six and taking it away from 1 to answer the question.*

Solution:

P (getting a six) $= \frac{10}{36} + \frac{1}{36} = \frac{11}{36}$

P (not getting a six) $= 1 - \frac{11}{36} = \frac{25}{36}$

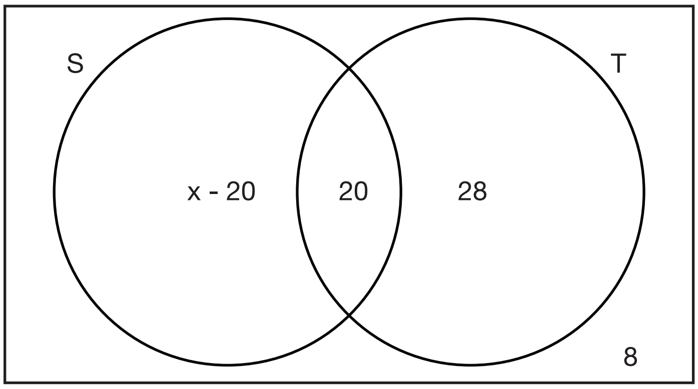
Example 4	**Teaching notes**
Two events, A and B, are complementary and make up the entire sample space. Also, $P(A') = 0,35$. a) Complete the statement $P(A) + P(B) = \dots$ b) Write down the value of $P(A\ and\ B)$ c) Write down the value of $P(B)$	Ask: *What is true of complementary events?* (If one occurs, the other cannot occur; at least one of the events must occur – in other words $P(A\ or\ B) = 1$; Both events together make up the entire sample space). Ask: *Are all complementary events mutually exclusive?* (Yes, there is no intersection in complementary events).
Solution: a) $P(A) + P(B) = 1$ b) $P(A\ and\ B) = 0$ c) $P(B) = P(A') = 0,35$	

Example 5	**Teaching notes**
A survey was conducted among 150 learners in Grade 10 at a school to establish how many of them owned the following devices: smartphone (S) or tablet (T). The results were as follows: • 8 learners did not own either a smartphone or a tablet • 20 learners owned both a smartphone and a tablet • 48 learners owned a tablet • x learners owned a smartphone a) Represent the above information in a Venn diagram.	Ask: *How many sets are represented?* (Two – smartphone and tablet) Ask: *Are the sets inclusive?* Yes – there are learners who own both a smartphone and a tablet) Tell learners to draw the frame and the two overlapping circles. Ask: *Which part of the Venn diagram must always be completed first?* (The intersection) Tell learners to find the information that describes this and to fill it in. (20 learners owned both a smartphone and a tablet). Ask: *How many learners are left that own ONLY a smartphone and ONLY tablet?* ($x - 20$ for the smartphone and 28 for a tablet because 48 – 20 = 28). Fill this in then fill in the number of learners who own neither a smartphone nor a tablet outside of the sets.
b) How many learners owned a smartphone?	Ask: *What is the total number of learners in the survey?* (150)

c) Calculate the probability that a learner selected at random from this group: (i) owned only a smartphone (ii) owned at most one type of device.	Remind learners that the total amount of learners will be the sample space and therefore the denominator. Ask: *Give another way in which we could ask how many learners owned at most one type of device.* (How many learners did not own both types of devices). Tell learners that this could either be done using complementary events: (1 subtract the probability of owning both) or adding those that owned only 1 and those that owned neither and finding the probability.

Solution:

a)



b) $x - 20 + 20 + 28 + 8 = 150$

$\therefore x = 114$

$\therefore 94$ learners owned smartphones

c) (i) $P(S\ only) = \frac{94}{150}$

(ii) P(owning both) $= \frac{20}{150}$

P(owning at most 1) $= 1 - \frac{20}{150}$

$= \frac{130}{150} = \frac{13}{15}$

7. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
8. Give learners an exercise to complete with a partner.
9. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=bh3yZuVSUC4
(What is probability?)

https://www.youtube.com/watch?v=vGcmjINp1x8
(Probability word problems)

https://www.mathsisfun.com/data/probability-events-mutually-exclusive.html
(Explanation of mutually exclusive)

Venn diagrams:
https://www.youtube.com/watch?v=z5Gc1R3EbZs
(Introduction)

https://www.youtube.com/watch?v=MSw0JDbfNhA
(Venn diagram notation)

https://www.youtube.com/watch?v=xwK--rNDI9E
(b) Ask: *what do we know about the total learners in the survey* (150) (Examples)

Term 3, Topic 5, Lesson 2

TREE DIAGRAMS

Suggested lesson duration: 2 hours

A POLICY AND OUTCOMES

CAPS Page Number	38
Lesson Objectives By the end of the lesson, learners should be able to: ● draw and use tree diagrams to solve problems ● understand the concept of dependent and independent events.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 20 from the Resource Pack.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
2 3	254 256	3	268	4 (2, 6, 7, 8)	288	12.5	414	10.5	430

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CONCEPTUAL DEVELOPMENT

C

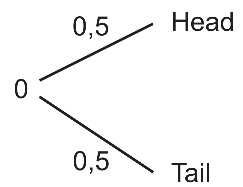
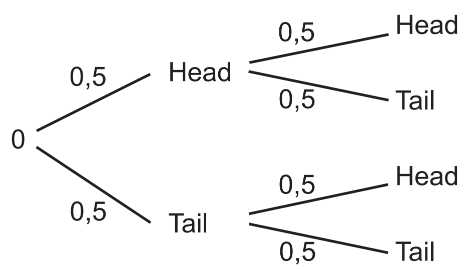
INTRODUCTION

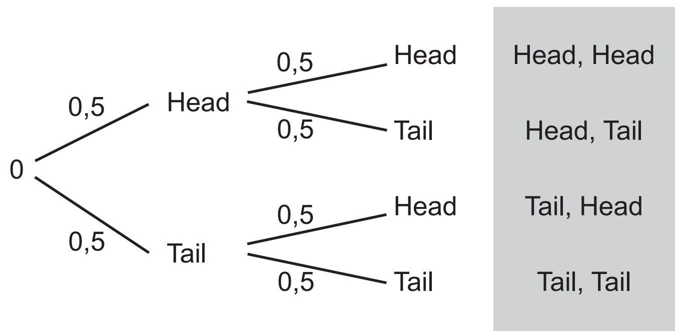
1. Learners have not studied tree diagrams in detail since Grade 9, so it is important that you revise the concepts.
2. Ensure that learners understand before moving on.

DIRECT INSTRUCTION

1. Give the following scenario to learners: *I am going to toss (flip) a coin twice. Let's draw a tree diagram to represent all the possible outcomes and all the probabilities*

   Ask: *What are the outcomes of tossing a coin?*
   (Getting heads or tails – two outcomes in total).
   Draw this tree diagram one piece at a time. Point out key issues to learners as you draw the tree diagram.

	Teaching notes
	Start at a point and draw 2 branches representing the two possible outcomes. Write the outcomes at the end of the branches. Write the theoretical probability ON the branch. Tell learners that they must not deviate from these conventions.
	Extend the tree diagram to represent the second toss Tell learners: *Even though there will only be one more toss, it needs to be represented twice – as if heads was tossed on the first throw and as if tails were tossed on the first throw – to cover ALL possibilities.*



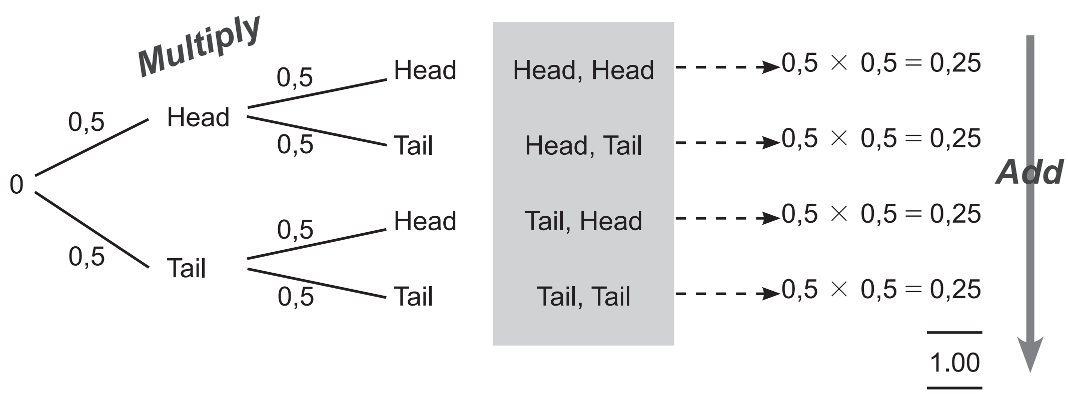
Write all the possible outcomes at the end of the branches.
Run your finger along the different branches of the tree diagram to show all the possibilities.

2. Tell learners: A tree diagram is a simple way of representing a sequence of events.

3. Point out that no matter how many branches come from one point, the probabilities on those branches should always add up to one because all the possibilities are represented.

4. Ask: *Are these events dependent or independent?*
   (Independent – how the coin landed the first time does not affect what will happen the second time)

5. Use the following diagram (add onto your final one) to show how probabilities are calculated using a tree diagram.



6. Ask learners to consider the probability of getting <u>at least one head</u>. Confirm the meaning of this statement. In the two tosses, one head will be acceptable but so will two heads as this also covers the statement 'at least one'.

7. Tell learners to look at all the outcomes.
   Ask: H*ow many of these outcomes have at least one head in them?*
   (Three of them do - only the last outcome has no heads at all).

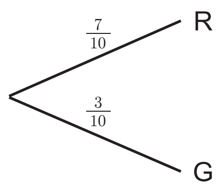
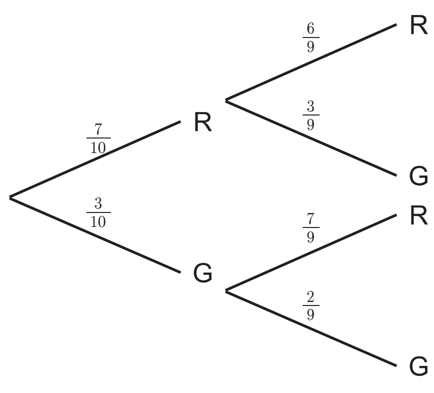
8. To calculate the probability of at least one head being tossed, we first multiply along the branches that lead to each of these three outcomes, then add all of those answers together.

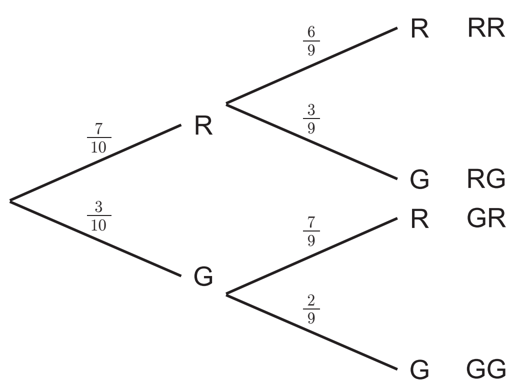
$$\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$$

9. Ask: *Describe a shorter way to find this probability which was discussed earlier.*
(1 subtract the probability of getting no heads at all as the two possibilities add up to 1 because those two outcomes – getting at least one head and getting no heads – represent all outcomes).

$$1 - \frac{1}{4} = \frac{3}{4}$$

10. Summarise the following points and ask learners to write them down in their exercise books:
    - To find the probability of something happening AND something else happening, multiply the probabilities together.
    - To find the probability of something happening OR something else happening, add up the probabilities.
11. Describe the following scenario to learners: *I have a bag with 7 red balls and 3 green balls in it. Without looking into the bag, I am going to take out one ball, and then another ball. The first ball will not be put back in the bag before the second one is drawn.*
12. Ask: Are these events independent or dependent?
(Dependent – what happens in the second draw depends on what happened in the first draw).
Draw the following tree diagram one piece at a time. Point out key issues as you draw.

	Teaching notes
	Start at a point and draw 2 branches representing the two possible outcomes. Write the outcomes at the end of the branches. Write the theoretical probability ON the branch. Tell learners that they must not deviate from these conventions.
	Extend the tree diagram to represent the second draw. Tell learners: *Even though there will only be one more draw, it needs to be represented twice – as if a red ball was drawn on the first draw and as if a green ball was drawn on the first throw - to cover ALL possibilities.* Ensure learners understand why the total balls has now changed to 9 and why the total reds went down by one when red was drawn first but did not go down one when green was drawn first.



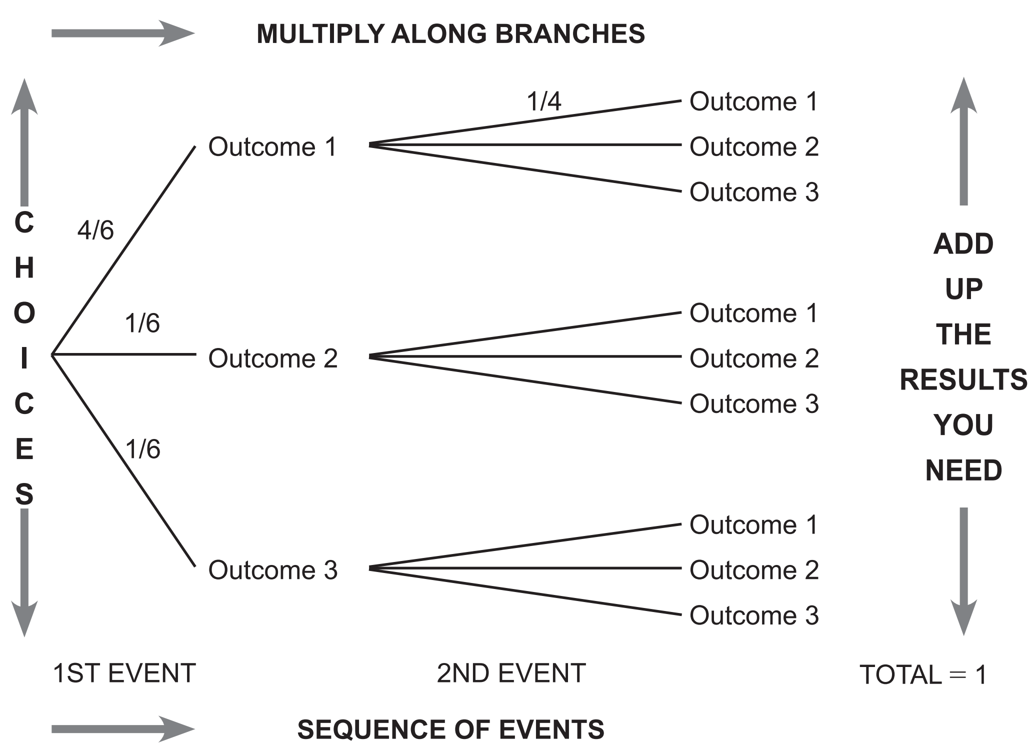
Write all the possible outcomes at the end of the branches.
Run your finger along the different branches to show all the possibilities.

13. Use the tree diagram from Point 12 to answer the questions:

	Teaching notes	**Answer**
How many possible outcomes are there? Name them.	The outcomes were listed at the end of the tree diagram. Remind learners that the end of each branch represents an outcome.	4 outcomes: RR RG GR GG
What is the probability of getting a red ball and then a green ball?	Say: *Look at the outcomes on the tree diagram – choose the outcome that is red first and green second. Now look at the probabilities along the branch that leads to this outcome. Multiply.*	$P(RG) = \frac{7}{10} \times \frac{3}{9}$ $= \frac{21}{90}$ $= \frac{7}{30}$
What is the probability of two colours the same?	Say: *Look at the outcomes on the tree diagram– choose the outcome that represent two colours the same. Now look at the probabilities along the branch that leads to these outcomes. Multiply along the branches and then add the possibilities.*	$P(RR) + P(GG)$ $= \left(\frac{7}{10}\right)\left(\frac{6}{9}\right) + \left(\frac{3}{10}\right)\left(\frac{2}{9}\right)$ $= \left(\frac{42}{90}\right) + \left(\frac{6}{90}\right)$ $= \frac{48}{90}$ $= \frac{8}{15}$ Point out why it was useful not to simplify until later – the fractions already had the same denominator when addition was required.

What is the probability of drawing a green ball in the second draw?	Say: *Look at the outcomes on the tree diagram – choose the outcomes that represent a green ball in the second draw.* *Now look at the probabilities along the branch that leads to these outcomes. Multiply along the branches and then add the possibilities.*	$P(RG)+P(GG)$ $=\left(\frac{7}{10}\right)\left(\frac{3}{9}\right)+\left(\frac{3}{10}\right)\left(\frac{2}{9}\right)$ $=\left(\frac{21}{90}\right)+\left(\frac{6}{90}\right)$ $=\frac{26}{90}$ $=\frac{13}{45}$
What is the probability of at least one green ball?	Ask: *What is another way of saying at least 1 green ball?* (Not two red balls). It is often useful to use the complementary rule when '*at least'* is used – there is only one outcome that doesn't have the 'at least one green ball'.	P (at least 1 G) $=1-\left(\frac{7}{10}\right)\left(\frac{6}{9}\right)$ $=1-\left(\frac{42}{90}\right)$ $=\frac{48}{90}$ $=\frac{8}{15}$

14. Summarise how all tree diagrams work by drawing the following diagram on the board for learners to write in their exercise books:
(A larger version of this diagram is available in the Resource Pack. Resource 20).

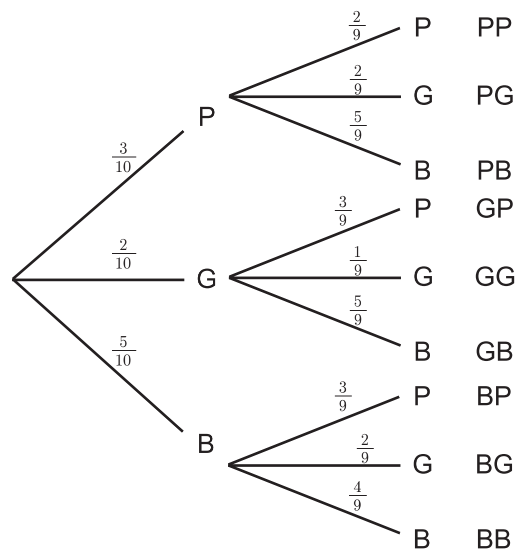


15. Do these worked examples from previous examinations. Tell learners to write them in their exercise books and make notes as they do the solutions with you.
Encouraging learners to: *'Tell me what to do'* before you do it on the board.

Example 1	**Teaching notes**
A packet of sweets contains 3 pink, 2 green and 5 blue sweets. Two sweets are removed in succession from the packet without replacing them. a) Draw a tree diagram to determine all possible outcomes.	Ask: *How many options are given?* (3) Say: *Look at the outcomes – choose those that represent 2 blue sweets (i) and those that represent a green and a pink sweet (ii).* *Now look at the probabilities along the branch that leads to these outcomes. Multiply along the branches and add the outcomes if there is more than one option.* Ask: *How many draws will be made/extensions needed?* (2) Learners need to plan for this, so that they don't run out of space. <u>Step 1</u>: Make the starting point quite far down the page, first going to three different outcomes. Once the outcomes have been written at the end of the branches, the probability of each should be written on the branch. <u>Step 2</u>: Extend the tree to the second sweet being drawn. Three branches must come from each of the three outcomes. Remind learners to be careful with their totals and confirm which colour sweet is being chosen at each option.
b) Determine the probability that: (i) both sweets are blue (ii) a green and a pink sweet are selected. (round to 3 decimal places) EC 2016	Say: *Look carefully at the outcomes – choose those that represent 2 blue sweets (i) and those that represent a green and a pink sweet (ii).* *Now look at the probabilities along the branch that leads to these outcomes. Multiply along the branches and add the outcomes if there is more than one option.*

Solution:

a)



b) (i) $P(BB) = \frac{5}{10} \times \frac{4}{9}$

$= \frac{20}{90}$

$= \frac{2}{9}$

$= 0{,}222$

(ii) $P(GP) + P(PG) = \left(\frac{2}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{3}{10}\right)\left(\frac{2}{9}\right)$

$= \left(\frac{6}{90}\right) + \left(\frac{6}{90}\right)$

$= \frac{12}{90}$

$= 0{,}133$

Example 2	**Teaching notes**
Paballo has a bag containing 80 marbles that are either green, yellow or red in colour. $\frac{3}{5}$ of the marbles are green and 10% of the marbles are yellow. Paballo picks two marbles out of the bag, one at a time and without replacing the first one.	
a) How many red marbles are in the bag?	This is not a probability question – but an arithmetic one. Ask: *How will we find the number of red marbles?* (Find $\frac{3}{5}$ of 80 and 10% of 80 so we know the number of green and yellow then subtract from 80).

b) Draw a tree diagram to represent the above situation.	Ask: *How many options are given?* (Three) *Ask: How many draws will be made/extensions needed?* (Two) As before learners need to plan carefully and think carefully when completing the probability on each branch.
c) What is the probability that Paballo will choose a green and a yellow marble? NSC NOV 2014	Say: *Look at the outcomes – choose those that represent a green and a yellow marble.* *Multiply along the branches leading to these outcomes and add.*

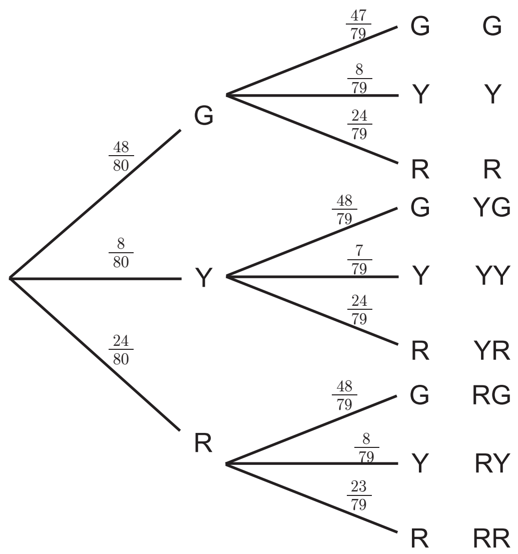
Solution:

a) $\frac{3}{5}$ of 80 $=$ 48

10% of 80 = 8

$\therefore$ there are 24 red marbles

b)

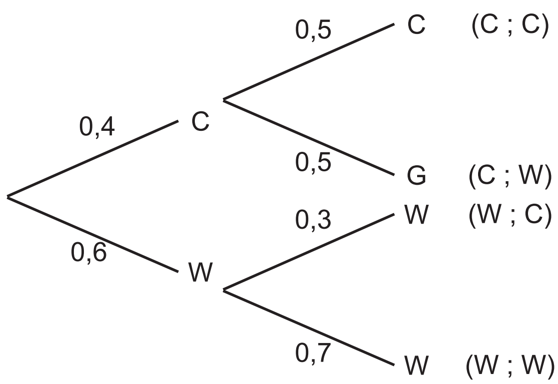


c)
$$P(GY)+P(YG)=\left(\frac{48}{80}\right)\left(\frac{8}{79}\right)+\left(\frac{8}{80}\right)\left(\frac{48}{79}\right)$$
$$=\left(\frac{384}{6320}\right)+\left(\frac{384}{6320}\right)$$
$$=\frac{768}{6320}$$
$$=\frac{48}{395}\ or\ 0{,}121$$

Example 3	**Teaching notes**
The probability that the first answer in a maths quiz competition will be correct is 0,4. If the first answer is correct, the probability of getting the next answer correct rises to 0,5. However, if the answer is wrong, the probability of getting the next answer correct is only 0,3. a) Represent the information on a tree diagram. Show the probabilities associated with each branch as well as the possible outcomes.	Tell learners that this may seem like a difficult question when it is first read. The instruction to draw a tree diagram is key because this will make it much clearer. Ask: *How many options are given?* (Two) Ask: *How many questions will be asked/extensions needed?* (Two) Ask: W*hat are the options?* (Correct or wrong)
a) Calculate the probability of getting the second answer correct. NSC NOV 2016	Say: *Look at the outcomes – choose those that represent the second answer being correct. Multiply along the branches leading to these outcomes and add.*

Solution:

a)



b) P(2nd answer correct)

$= P(C \text{ and } C) + P(W \text{ and } C)$

$= (0,4)(0,5) + (0,6)(0,3)$

$= 0,38$

16. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.

17. Give learners an exercise to complete with a partner.

18. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=mkDzmI7YOx0

https://www.youtube.com/watch?v=kNOrDWm15bY

https://www.youtube.com/watch?v=gI5OoCrinDs

Term 3, Topic 5, Lesson 3

VENN DIAGRAMS

Suggested lesson duration: 2 hours

POLICY AND OUTCOMES

A

CAPS Page Number	38
Lesson Objectives By the end of the lesson, learners should be able to: ● draw and use Venn diagrams to solve problems.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. Write work on the chalkboard before the learners arrive. For this lesson have the terms written and empty Venn diagrams drawn ready to be shaded.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
4	261	2	261	1 4 (4)	277 288	12.2 12.3 12.7	401 404 422	10.2 10.4	411 424

C

CONCEPTUAL DEVELOPMENT

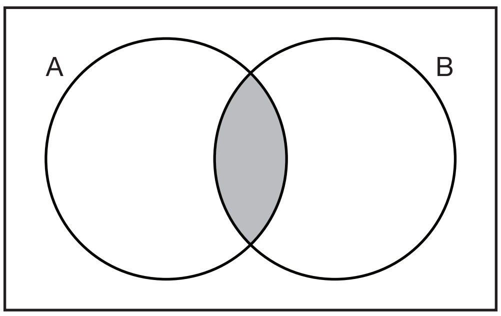
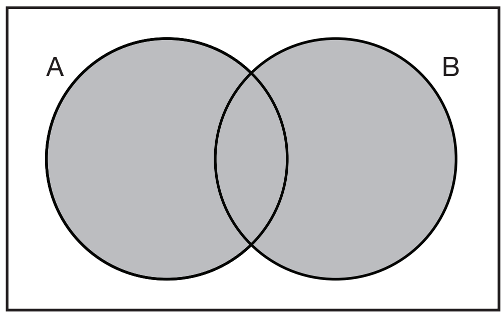
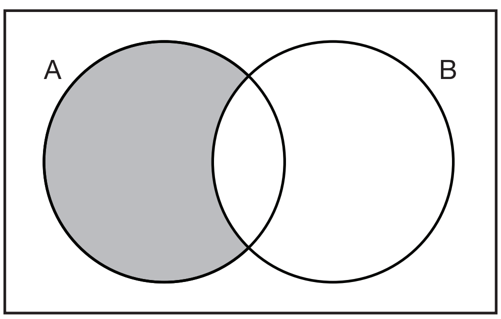
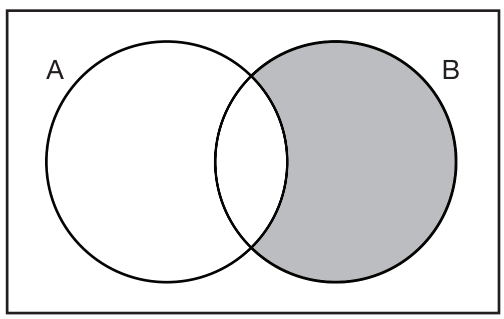
INTRODUCTION

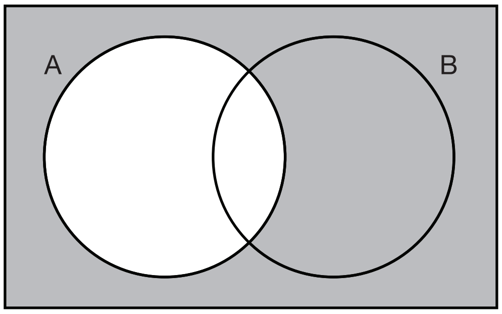
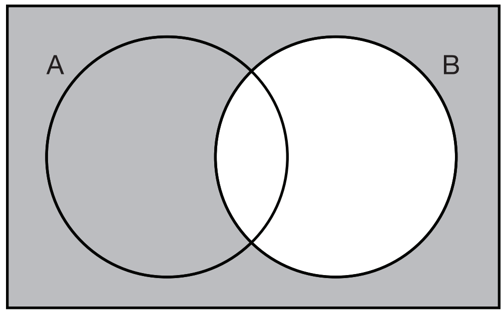
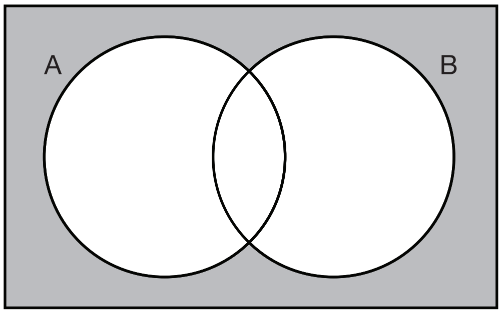
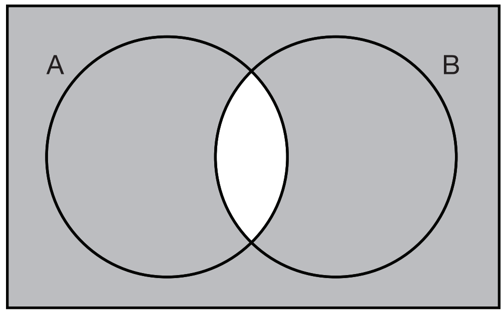
1. A good understanding of Venn diagrams is important in being successful in this section. Learners need to be able to draw them from given information and to read information correctly from them when given the diagram.

DIRECT INSTRUCTION

1. Remind learners that the learned about Venn diagrams in Grade 10.

2. Use the table below to summarise and revise Venn diagram concepts for learners.
   Start with the word, ask learners what it means and then ask for a volunteer to shade in the appropriate areas before learners take it down in their exercise books.

Intersection (A and B)	Union (A or B)
	
A only	**B only**
	

not A	not B
	
not (A or B)	not (A and B)
	

3. Use the table to discuss the symbols used for some of these terms.
   Add each symbol as you discuss it and tell learners to do the same.

intersection	union
$A \cap B$	$A \cup B$
A and B	A or B

not A	not B	not (A or B)
A'	B'	$(A \text{ or } B)'$

4. Remind learners of the identity linked to Venn diagrams:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

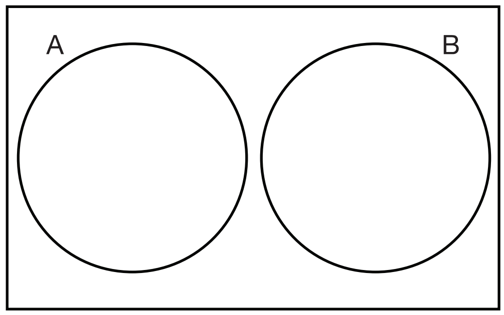
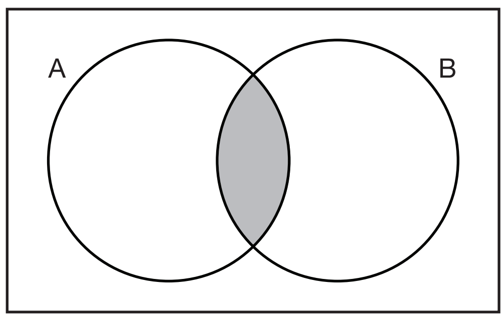
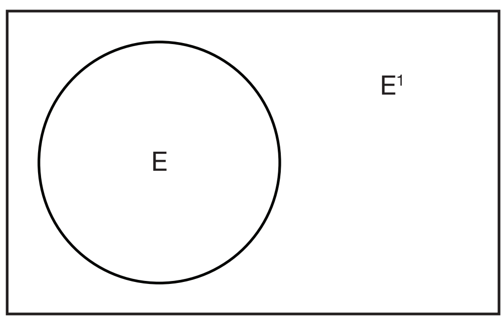
   Ask: *Why do we need to subtract the intersection after adding the probabilities of the two sets?*
   (If we added the probability of set A to set B, the intersection would have been covered twice).

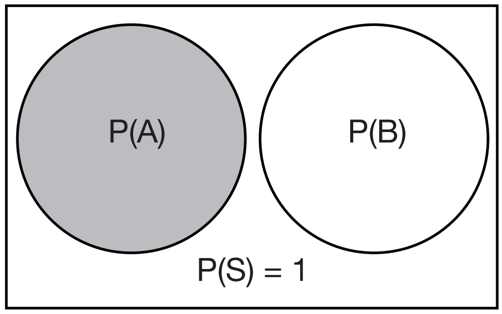
5. Discuss the following terms with learners.

   For each term, ask: *What does this mean?* Ask for a volunteer to draw a Venn diagram representing each concept.

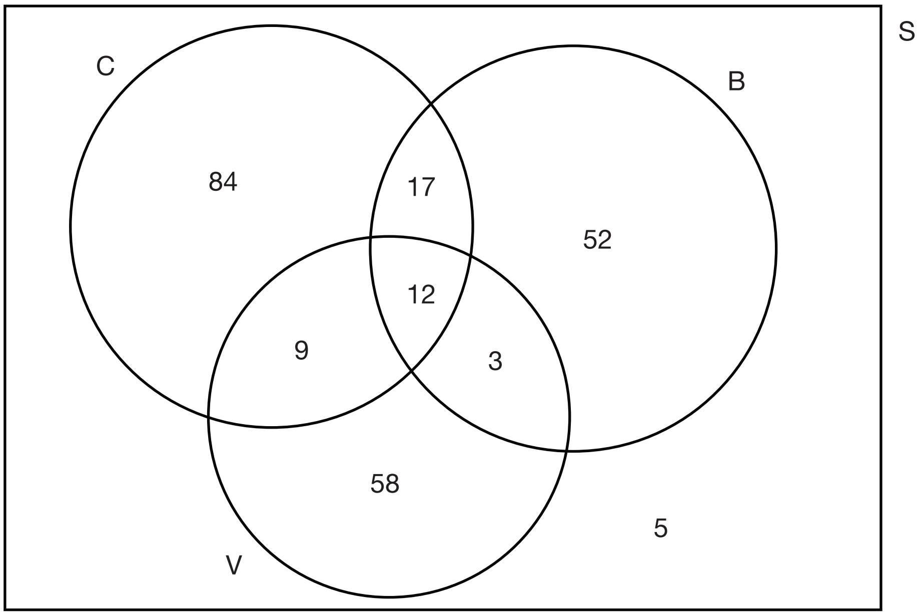
   - Mutually exclusive events
   - Inclusive events
   - Complementary events
   - Exhaustive events.

6. Where possible, use learners responses and diagrams to summarise for the learners and ask the learners to write the summary in their exercise books.

Term	**Explanation and example**	**Venn diagram**
Mutually exclusive events	Events with no intersection. Example: Boys and girls from a class.	
Inclusive events	Events with an intersection. Example: Learners that take mathematics and learners that take History.	
Complementary events	Events that contain all the possible outcomes between them. Example: A learner will either pass a test or fail a test. Only one event can happen at once but one must happen.	

Exhaustive events	Events where all possibilities from the sample space are covered by the two events. There can be an intersection. There are no elements outside the two sets.	

7. Learners will be expected to work with three-way Venn diagrams this year. Draw the following Venn diagram on the board from the 2014 final examination and use it to explain:



Tell learners that the situation represents:

240 customers were surveyed at a fast food outlet. The diagram shows the number of customers who bought cheese burgers (C), bacon burgers (B) and vegetarian burgers (V).

Before discussing probability, ask a few questions to ascertain whether learners understand the Venn diagram.

How many customers:

- *Didn't buy any of the three burgers mentioned?* (5)
- *Bought all three types of burgers mentioned?* (12)
- *Bought ONLY cheese burgers?* (84)
- *Bought cheese burgers and bacon burgers?* (29 → 17 + 12)
- *Bought vegetarian burgers and bacon burgers?* (15 → 12 + 3)
- *Bought vegetarian burgers?* (82 → 58 + 9 + 12 + 3)

Use the diagram for each question. Use your hands to show which part of the Venn diagram gives each answer.

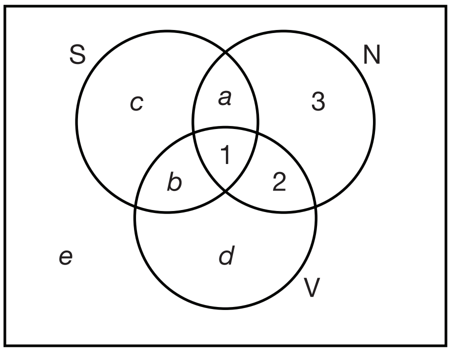
Point to some of the numbers in the Venn diagram and ask: What do these people represent?

Examples: The '9' represents people that bought cheese and vegetarian but not bacon burgers.
The '52' represents people that bought only bacon burgers.

Do the following probability questions with learners. Remind learners that the total number of people surveyed will always be the denominator and the numbers similar to what you were asking previously will be the numerator. Do not simplify the fractions – the aim is to confirm learners know where the numbers to be used in the fractions come from.

If a customer from this group is selected at random, determine the probability that he/she:	
bought only a vegetarian burger	$\frac{58}{240}$
bought a cheese burger and a bacon burger	$\frac{29}{240}$
did not buy a cheese burger	$\frac{52+3+58+5}{240}=\frac{118}{240}$
bought a bacon burger or a vegetarian burger	$\frac{52+17+12+3+9+58}{240}=\frac{151}{240}$

8. Ask whether there are any questions before you move on.
9. Do some worked examples from previous examinations. Learners must write them in their exercise books and make notes as they work.

<table>
<tr><th>Example 1</th><th>Teaching notes</th></tr>
<tr><td>A survey was conducted amongst 100 learners at a school to establish their involvement in three codes of sport, soccer, netball and volleyball. The results are shown below:
• 55 learners play soccer (S)
• 21 learners play netball (N)
• 7 learners play volleyball (V)
• 3 learners play netball only
• 2 learners play soccer and volleyball
• 1 learner plays all three sports
The Venn diagram below shows the information above.

</td><td>This question is very similar to being asked to draw the Venn diagram from the information given. They will still need to work out what numbers go in each section.
Ask: Where should we always start when completing a Venn diagram?
(The intersection)
Say: Then we need to work our way outwards from there until the last areas filled in are those that represent ONLY a certain set.
Tell learners to note that the 1 has been filled in.</td></tr>
<tr><td>a) Determine the values of a, b, c, d and e.</td><td>Ask: How will we find the value of ‘a’?
(It is the only missing value, so knowing that 21 learners play netball can be used and the other numbers subtracted).
Ask: How will we find the value of ‘b’?
(It forms part of the intersection of soccer and volleyball and we know that is 2 but 1 has already been counted).
Say: c and d can now both be found using the totals for each sport and e will be found in a similar way – we know the total surveyed.</td></tr>
<tr><td>b) What is the probability that one of the learners chosen at random from this group plays netball or volleyball?
NSC NOV 2016</td><td>Ask a learner to show you on the Venn diagram which areas represent netball OR volleyball. Remind learners that or is more.
(Only c and e are excluded).</td></tr>
</table>

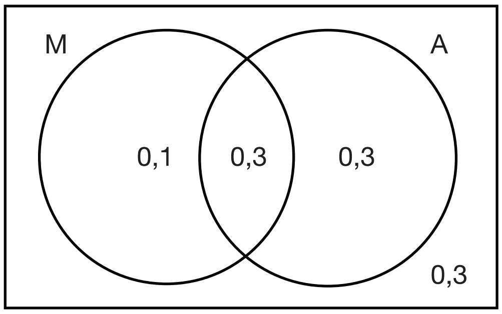
Solution:

a) $a = 15; b = 1; c = 38; d = 3; e = 37$

b) $P(N\ or\ V) = \frac{25}{100} = \frac{1}{4}$

Example 2	**Teaching notes**
William writes a Mathematics examination and an Accounting examination. He estimates that he has a 40% chance of passing the Mathematics examination. He estimates that he has a 60% chance of passing the Accounting examination. He estimates that he has a 30% chance of passing both. Determine the probability that William will fail Mathematics and Accounting. EC 2015	Point out that this question does not give a suggestion as to what method could be used to assist in calculating the answer. The key word is 'both'. In other words, an intersection which in turn implies that a Venn diagram may be useful. Ask: *How many sets are represented?* (Two – Mathematics and Accounting). Ask: *Are they inclusive?* (Yes – there is a chance of passing both). Tell learners to draw the frame and the two overlapping circles. Ask: *Which part of the Venn diagram must always be completed first?* (The intersection). Tell learners to find the information that describes this (30% chance of passing both) and to fill it in. Ask: *How much chance is still left after the intersection has been completed of passing each exam?* (10% for Mathematics and 30% for Accounting). *Fill this in then finally check if it totals to 100% - if not there is something outside the sets (in other words it is not exhaustive).* Failing both will be the probability on the outside of the sets.

Solution:



P (fail both) $= 0,3$

10. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
11. Give learners an exercise to complete with a partner.
12. Walk around the classroom as learners do the exercise. Support learners where necessary.

ADDITIONAL ACTIVITIES/ READING

D

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=IqaBt1_6PDA

https://www.youtube.com/watch?v=ORj2C_R27WU

https://www.youtube.com/watch?v=YX34Xh0Dra8

Term 3, Topic 5, Lesson 4

CONTINGENCY TABLES

Suggested lesson duration: 1,5 hours

A POLICY AND OUTCOMES

CAPS Page Number	38
Lesson Objectives By the end of the lesson, learners should be able to: ● use contingency tables to solve problems ● understand the concept of dependent and independent events.	

B CLASSROOM MANAGEMENT

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 21 from the Resource Pack.
5. Write work on the chalkboard before the learners arrive. For this lesson draw the table for point 1.
6. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
5	264	4	273	4 (1, 3)	288	12.6 12.8	417 425	10.6	424

CONCEPTUAL DEVELOPMENT

C

INTRODUCTION

1. Learners need to be able to read from a contingency table in order to answer many probability questions. Learners often find it difficult.
2. Go through as many examples as possible and ask many learners to answer your questions or show you where the answer would be found on the table.

DIRECT INSTRUCTION

1. Describe a contingency table. Use this table as an example:

		Body Image			
		About Right	Overweight	Underweight	**Total**
Gender	Female	560	263	37	760
	Male	395	72	73	440
	Total	855	235	110	1200

2. Say: *A contingency table shows the distribution of one variable in rows and another in columns. It is used to study the correlation between the two variables*.
3. Before discussing the table and what can be read from it, ask learners to copy the definition and the table into their exercise books.
4. Look at the table in more detail.
   Ask: *What was the survey about and who participated?*
   (The survey was about how people felt about their body image. Males and females were questioned, and their answers were listed separately).

   Ask:
   - *How many females were surveyed?* (760)
   - *How many people were surveyed altogether?* (1200)
   - *How many people in total consider themselves underweight?* (110)
   - *How many of those people are male?* (73)
   - *How many males consider themselves overweight? (*72)
   - *How many people in total think their weight is just right?* (855).

Stop after each question to confirm where the answer was found on the table

5. Contingency tables can be used when calculating probabilities. Use the same table to work through some probability questions:

> Although the contingency table shows that it is what people think of their bodies rather than what they actually are, the following questions were easier asked as if the people were underweight or overweight.
> There is no need to simplify the fractions. The important idea here is to know where the numbers are being read for both the numerator and denominator.

	Teaching notes	
Find the probability that a person chosen at random is:		
male	Each of these questions has a focus on one of the totals of the various headings (other than the grand total which will be the denominator).	$\frac{440}{1200}$
overweight		$\frac{235}{1200}$
happy with their weight		$\frac{855}{1200}$
female		$\frac{760}{1200}$
underweight		$\frac{110}{1200}$
a female who thinks she is overweight	Each of these questions has a focus on a combination of two of the headings (other than the grand total which will be the denominator).	$\frac{163}{1200}$
a male who thinks he is underweight		$\frac{73}{1200}$
a female who is happy with her weight		$\frac{560}{1200}$
an overweight person who is male		$\frac{72}{1200}$
underweight, given that they are female	Each of these questions has a focus on a different area for the total since the question states, 'given that'. This means that a particular group is already chosen. For example, given that they are female means that the group of females are the new sample space.	$\frac{37}{760}$
overweight, given that they are male		$\frac{72}{440}$
male, given that they are happy with their weight.		$\frac{295}{855}$
female, given that they are underweight		$\frac{37}{110}$
male, given that they are overweight		$\frac{72}{235}$

6. Ask whether there are any questions.

7. Do some worked examples from previous examinations. Learners must write them in their exercise books and make notes as they work.

> The table has been enlarged and is available in the Resource Pack. Resource 21.

Example 1

The following contingency table shows information on the drivers' tests of 100 drivers tested at a test centre in Port Elizabeth.

	Male	Female	Total
Pass	30	47	77
Fail	7	16	23
Total	37	63	100

A driver is randomly selected from the 100 drivers.

a) Determine the probability that a female that failed is selected.

Teaching notes

Ask: *What total is being considered?*
(The grand total 100)
Ask: *Which two groups do we need to focus on?*
(Female and failed – 16).

b) Determine the probability that the driver passed, given that it is a male.

EC 2015

Teaching notes

Ask: W*hat total is being considered?*
(The total males – 37).
Ask: *Which two groups do we need to focus on?*
(Male and passed – 30).

Solution:

a) $\frac{16}{100} = \frac{4}{25}$

b) $\frac{30}{37}$

Example 2

A survey was conducted amongst 60 boys and 60 girls in Grade 8 relating to their participation in sport. 20 girls did not participate in any sport and 50 boys did participate in a sport.

a) Complete a two-way contingency table for the above survey.

Teaching notes

Ask: *What are all the categories?*
(Girls, boys, do sport, don't do sport).
Tell learners to draw up the table with the above headings.
Once that is done, tell learners to fill in the numbers.

b) What is the probability that if a Grade 8 pupil is chosen at random that: (i) it is a girl and participates in sport? (ii) the pupil does not participate in sport and is not female? EC 2016	(i) Ask: *What total is being considered?* (The grand total – 120). Ask: *Which two groups do we need to focus on?* (Girl and does sport – 40). (ii) Ask: *What total is being considered?* (The grand total – 120). Ask: *Which two groups do we need to focus on?* (Boy and doesn't do sport – 10).

Solution:

a)

	Boys	Girls	
Sport	50	40	90
No sport	10	20	30
	60	60	120

b) $P(GS)=\frac{40}{120}=\frac{1}{3}$

c) $P(B\,NS)=\frac{10}{120}=\frac{1}{12}$

8. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
9. Give learners an exercise to complete with a partner.
10. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=D9XbNd3gikE

https://www.youtube.com/watch?v=0oCoc5B1IVU

Term 3, Topic 5, Lesson 5

DEPENDENT AND INDEPENDENT EVENTS

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	38
Lesson Objectives By the end of the lesson, learners will be able to • explain the concepts of dependent events and independent events and prove whether events are independent or not.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. You will need Resource 21 from the Resource Pack.
5. The table below provides references to this topic in Grade 11 textbooks. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
						12.4 12.5	411 412	10.3	418

C CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. A general understanding of dependent and independent events is useful in daily life. This understanding can assist learners in taking the concept a step further and using it in probability calculations.

DIRECT INSTRUCTION

1. Start the lesson by explaining the difference between dependent and independent events. Use real life examples at first.
   A dependent event is an event that is influenced by another event.
   For example, if you don't top up your electricity pay-as-you-go, there will be no electricity.
   An independent event is an event that has no connection with another event.
   For example, owning a pet and wearing a blue shirt.
   The event has no effect on the probability of another event occurring.

2. Use the following mathematical scenario to illustrate these events further. Learners should take these down.
   Remind learners that balls are drawn from the bag without looking – they are randomly drawn.
   A bag contains 4 red, 3 green and 2 blue balls. 2 balls are drawn from the bag.

   Example 1:

   After the first ball is drawn, it is replaced. The second ball is drawn. What is the probability of getting a blue then a green ball?

$$P(BG) = \frac{2}{9} \times \frac{3}{9} = \frac{6}{81} = \frac{2}{27}$$

Remind learners: *As the ball was replaced the sample space remained the same size.*

   Example 2:

   After the first ball it is not replaced. What is the chance of getting a blue ball then a green ball?

$$P(BG) = \frac{2}{9} \times \frac{3}{8} = \frac{6}{72} = \frac{1}{12}$$

Remind learners: *As the ball was not replaced the sample space needed to be adjusted.*

3. Ask: *Which example represents an independent event? Why?*
   (The first, because the ball is replaced, it had no effect on the second drawing of a ball)

4. Tell learners to notice how the probability was calculated for the first example (the independent events). Probability was calculated by multiplying the probability of one event (the blue ball) by the probability of the other event (the green ball).
   Say: *This idea is used to decide if two events are independent or not. If the probability of one event* <u>*and*</u> *another event is the same as the product of the probabilities of those events, then the events are independent.*

5. Write the following on the board:

$$P(A\ and\ B) = P(A) \times P(B)$$

   Say: *This is always true if events are independent of each other.*

6. Ask whether there are any questions.

7. Do some worked examples from previous examinations. Learners must write them in their exercise books and make notes as they work.

The table has been enlarged and is available in the Resource Pack. Resource 21.

Example1	**Teaching notes**
Given: $P(A) = 0,2$ $P(B) = 0,5$ $P(A\ or\ B) = 0,6$ Where A and B are two different events. a) Calculate $P(A\ and\ B)$	Ask: *How will we find* $P(A\ and\ B)$*?* Use the identity $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$
b) Are the events A and B independent? Show your calculations. NSC NOV 2016	Ask: *What calculation will be needed to prove independence?* $P(A\ and\ B) = P(A) \square P(B)$
Solution: a) $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$ $0,6 = 0,2 + 0,5 - P(A\ and\ B)$ $-0,1 = -P(A\ and\ B)$ $\therefore P(A\ and\ B) = 0,1$ b) $P(A\ and\ B) = 0,1$ $P(A) \times P(B) = 0,2 \times 0,5 = 0,1$ $\therefore$ the events are independent	

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<table>
<tr><th>Example 3</th><th>Teaching notes</th></tr>
<tr><td>The table below shows data on the monthly income of employed people in two residential areas. Representative samples were used in the collection of the data.

<table>
<tr><th>MONTHLY INCOME (IN RANDS)</th><th>AREA 1</th><th>AREA 2</th><th>TOTAL</th></tr>
<tr><td>$x < 3200$</td><td>500</td><td>460</td><td>960</td></tr>
<tr><td>$3\,200 \leq x < 25\,600$</td><td>1182</td><td>340</td><td>1522</td></tr>
<tr><td>$x \geq 25\,600$</td><td>150</td><td>14</td><td>164</td></tr>
<tr><td>Total</td><td>1832</td><td>814</td><td>2646</td></tr>
</table>
</td><td>Point out to learner that this question is a combination of reading from contingency tables and answering a question on independent events.</td></tr>
<tr><td>a) What is the probability that a person chosen randomly from the entire sample will be:
(i) from Area 1
(ii) from Area 2 and earn less than R3200 per month
(iii)a person from area 2 who earns more than or equal to R3200</td><td>Ask: What total is being considered?
(i) the grand total, 2646
(ii) the grand total, 2646
(iii) the grand total, 2646
Ask: Which two groups do we need to focus on?
(i) Area 1 and its total
(ii) Area 2 and less than 3200
(iii) Area 2 and more than 3200</td></tr>
<tr><td>b) Prove that earning an income of less than R3200 per month is not independent of the area in which a person resides.</td><td>Ask: What calculation will be needed to prove dependence?
$P(A \; and \; B) \neq P(A) \times P(B)$</td></tr>
<tr><td>c) Which is more likely: a person from Area 1 earning less than R3200 or a person from Area 2 earning less than R3200?
Show calculations to support your answer.
NSC NOV 2015</td><td>Ask: What do we need to find the probabilities of in order to compare them?
(Less than 3200 given a person is from Area 1 AND less than 3200 given that a person is from Area 2).</td></tr>
</table>

Solution:

a) (i) $P(Area1) = \frac{1832}{2646} = 0{,}692$

(ii) $P(A2\ and < 3200) = \frac{460}{2646} = 0{,}174$

(iii) $P(A2\ and > 3200) = \frac{354}{2646} = 0{,}134$

b) $P(A1) \times P(< 3200) = \frac{1832}{2646} \times \frac{960}{2646} = 0{,}251$

$P(A1\ and < 3200) = \frac{500}{2646} = 0{,}189$

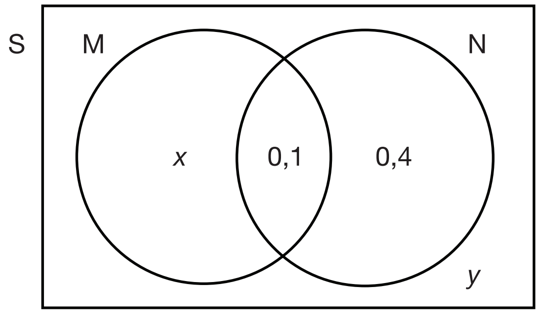
$\therefore P(A1\ and < 3200) \neq P(A1) \times P(< 3200)$

∴ earning an income of less than R3200 per month is not independent of the area in which a person resides.

c) $P(A1\ and < 3200) = \frac{500}{1832} = 0{,}273$

$P(A2\ and < 3200) = \frac{460}{814} = 0{,}565$

A person from Area 2 earning less than R3200 is more likely.

Example 4	Teaching notes
The Venn diagram below shows two independent events, M and N.  Determine the values of probabilities x and y. Show all calculations. EC 2015	Ask: *If the events are independent, what will be true?* $(P(A\ and\ B) = P(A) \times P(B))$ Tell learners to use this, substitute what is known and find x. Ask: *Once x has been found, how can we find y?* (Use the fact that all probabilities in the sample space must equal 1).

Solution:

$P(M\ and\ N) = P(M) \times P(N)$

$0{,}1 = P(M) \times 0{,}5$

$P(M) = 0{,}2$

$\therefore x = 0{,}1$

$0{,}1 + 0{,}1 + 0{,}4 + y = 1$

$\therefore y = 0{,}4$

8. Ask directed questions so that you can ascertain learners' level of understanding. Ask learners if they have any questions.
9. Give learners an exercise to complete with a partner.
10. Walk around the classroom as learners do the exercise. Support learners where necessary.

D ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=jos1yBC_L8E

https://www.youtube.com/watch?v=gNRT2KoyT7U

https://www.youtube.com/watch?v=AJKRq51N0ic

Term 3, Topic 5, Lesson 6

REVISION AND CONSOLIDATION

Suggested lesson duration: 1 hour

POLICY AND OUTCOMES

A

CAPS Page Number	38
Lesson Objectives By the end of the lesson, learners will have revised: • all concepts covered in previous lessons.	

CLASSROOM MANAGEMENT

B

1. Make sure that you are ready and prepared.
2. Advance preparation: Work through the lesson plan and exercises.
3. Write the lesson heading on the board before learners arrive.
4. The table below provides references to this topic in Grade 11 textbooks. Some books had exercises on work not covered directly in the previous lessons. As any practice is useful, those exercises have been included here. Plan when you will get learners to practice the concepts learned by completing the exercises. Work through the lesson plans and decide where you will get learners to do the exercises. Indicate this on your lesson plans.

LEARNER PRACTICE

MIND ACTION SERIES		PLATINUM		VIA AFRIKA		CLEVER		EVERYTHING MATHS (SIYAVULA)	
EX	PG	EX	PG	EX	PG	EX	PG	EX	PG
Rev	265	Rev	275	2 3 Qu's Ch 11	280 283 290 294	Rev	426	10.7	436

C

CONCEPTUAL DEVELOPMENT

INTRODUCTION

1. Ask learners to revise what they have learned in this section. Spend time pointing out issues that you know are important as well as problems that you encountered from your own learners.
2. If learners want you to explain a concept again, do that now.

DIRECT INSTRUCTION

1. Ask learners to do the revision exercise from their textbook. If you have an extra worksheet or a past test paper, this would also be an excellent way for them to consolidate what they have learned. It would also give them the opportunity of knowing what to expect when they must do an assessment.
2. Walk around the classroom as learners do the exercise. Support learners where necessary.

D

ADDITIONAL ACTIVITIES/ READING

Further reading, listening or viewing activities related to this topic are available on the following web links:

https://www.youtube.com/watch?v=mdYilUV7GwQ

https://www.youtube.com/watch?v=gtCFMsxp6ag